The Bilateral Hilbert Space

$\mathbb{CP}^\infty$ as the Space of Crossing Positions, the Born Rule from the Fubini–Study Metric, and the Koide Formula as a Transition Probability

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A Philosophy of Time, Space and Gravity

`ontologia.co.uk`

March 29, 2026

Abstract

We identify the bilateral Hilbert space with $\mathbb{CP}^\infty$, the space of all crossing directions of ∞_0. A quantum state $|\psi\rangle$ is a ray in $\mathbb{CP}^\infty$ — a specific crossing direction. The wave function $\psi = |\psi|e^{i\phi}$ is a section of the tautological line bundle γ^1 over $\mathbb{CP}^\infty$, with amplitude $|\psi|$ the egress face and phase $e^{i\phi}$ the ingress face. The Born rule $P = |\langle\psi|\phi\rangle|^2$ is the natural probability measure from the Fubini–Study metric on $\mathbb{CP}^\infty$ — not an axiom but a geometric consequence. The Heisenberg commutator $[x,p] = i\hbar$ follows from the curvature of γ^1: the symplectic form ω_{FS} evaluated on the egress (position) and ingress (momentum) directions gives $\omega_{\mathrm{FS}} = \hbar$. The Koide formula is identified as a Fubini–Study transition probability: $K_n = \cos^2(\theta_n)$ where $\tan\theta_n = 1/\sqrt{n}$, giving the sequence $K_n = n/(n+1)$. The four bilateral Koide values $\{1/2, 2/3, 3/4, 4/(3\varphi)\}$ are the Fubini–Study probabilities for $n = 1, 2, 3$ (free particles) and the confined modification of $n = 4$ (up-type quarks), with the confinement factor $5/(3\varphi) = p_{\mathcal{T}}/(\dim_{\mathbb{R}}(\mathbb{CP}^2) \times \varphi)$ appearing identically in the chiral condensate. $\mathbb{CP}^2$ is the finite truncation of $\mathbb{CP}^\infty$ used for the Standard Model; $\mathbb{CP}^\infty$ is the full bilateral Hilbert space from which quantum mechanics emerges.

Contents

1 The Bilateral Hilbert Space

Definition 1 (Bilateral Hilbert Space)**.** *The bilateral Hilbert space is:*

$$\mathcal{H}_{\text{bilateral}} = \mathbb{CP}^\infty, \tag{1}$$

the infinite-dimensional complex projective space — the space of all rays (complex lines through the origin) in an infinite-dimensional complex Hilbert space $\mathbb{C}^\infty$.

Proposition 1 ($\mathbb{CP}^\infty$ from ∞_0)**.** $\mathbb{CP}^\infty$ *is the space of all crossing directions of* ∞_0.

Proof. ∞_0 is the ground state — non-dimensional, non-directional, prior to all structure [1]. It has no preferred crossing; all crossings are equally valid. A crossing selects one direction out of all possible directions. The space of all directions in an infinite-dimensional complex vector space is $\mathbb{CP}^\infty$. A quantum state $|\psi\rangle$ is a ray in $\mathbb{CP}^\infty$ — a specific crossing direction of ∞_0. □

Remark 1. $\mathbb{CP}^\infty$ *is the classifying space* $BU(1)$ *for complex line bundles. Its cohomology ring is* $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[c_1]$ *— a polynomial ring generated by the first Chern class* c_1. *This is the algebraic backbone of the bilateral Hilbert space: every quantum observable corresponds to a cohomology class of* $\mathbb{CP}^\infty$.

2 The Wave Function as a Section of γ^1

$\mathbb{CP}^\infty$ carries a canonical complex line bundle: the tautological bundle γ^1. Over each point $[\psi] \in \mathbb{CP}^\infty$ (a ray), the fibre of γ^1 is the line represented by $[\psi]$ itself.

Proposition 2 (Wave Function as Section)**.** *The wave function* $\psi = |\psi|e^{i\phi}$ *is a section of* γ^1 *over* $\mathbb{CP}^\infty$*:*

- *The amplitude* $|\psi|$ *is the* egress face *— the length of the vector in the fibre line, the observable part.*
- *The phase* $e^{i\phi}$ *is the* ingress face *— the angle within the fibre line, the potential part.*

This is precisely the Wavefunction^2 interpretation [2]: ψ is a bilateral object with both faces simultaneously present. Standard quantum mechanics reads only the amplitude $|\psi|$; the bilateral framework reads the complete section.

3 The Born Rule from the Fubini–Study Metric

$\mathbb{CP}^\infty$ carries a canonical Riemannian metric: the Fubini–Study metric g_{FS}. The geodesic distance between two states $|\psi\rangle$ and $|\phi\rangle$ is:

$$d_{\text{FS}}(|\psi\rangle, |\phi\rangle) = \arccos |\langle\psi|\phi\rangle|. \tag{2}$$

Theorem 3 (Born Rule from Fubini–Study)**.** *The transition probability between two states is the Fubini–Study transition probability:*

$$P(|\psi\rangle \to |\phi\rangle) = \cos^2\!\big(d_{\text{FS}}(|\psi\rangle, |\phi\rangle)\big) = |\langle\psi|\phi\rangle|^2. \tag{3}$$

The Born rule is not an axiom. It is the natural probability measure on $\mathbb{CP}^\infty$ *derived from the Fubini–Study metric.*

Proof. The Fubini–Study metric is the unique unitarily invariant metric on $\mathbb{CP}^\infty$ (up to scale). The natural probability of transitioning from one state to another is therefore $\cos^2(d_{\mathrm{FS}})$, which equals $|\langle\psi|\phi\rangle|^2$ by the definition of d_{FS}. □

The bilateral interpretation: the Fubini–Study distance IS the bilateral crossing angle. The probability IS $\cos^2$ of the crossing angle. This is Malus's law — the probability that a photon passes a polariser at angle θ is $\cos^2\theta$ — which Wavefunction^2 identifies as the bilateral Born rule [2].

4 The Heisenberg Commutator from Curvature

The Fubini–Study metric on $\mathbb{CP}^\infty$ is Kähler: it comes with a compatible complex structure and symplectic form ω_{FS}. The tautological bundle γ^1 has curvature $F_{\gamma^1} = -2\pi i\,\omega_{\mathrm{FS}}$, and first Chern class $c_1(\gamma^1) = [\omega_{\mathrm{FS}}/(2\pi)]$.

Theorem 4 (Heisenberg Commutator from Curvature). *The Heisenberg commutator $[x,p] = i\hbar$ follows from the Fubini–Study symplectic form evaluated on the bilateral egress (position) and ingress (momentum) directions:*

$$[x,p] = i\hbar \cdot \omega_{\mathrm{FS}}(\partial_x, \partial_p) = i\hbar. \tag{4}$$

Proof. In the bilateral framework [3]: position x is the τ-face (egress, actualised), momentum p is the spin-face (ingress, potential). They are the two faces of the same bilateral crossing. The non-commutativity $[x,p] \neq 0$ arises because measuring x first (selecting the egress face) then p differs from measuring p then x by exactly one bilateral face-crossing.

In geometric quantisation, the prequantum line bundle over the classical phase space is γ^1 with connection ∇ of curvature $F = -2\pi i\,\omega_{\mathrm{FS}}$. The quantisation procedure gives operators $\hat{f}$ from classical observables f with commutator:

$$[\hat{f}, \hat{g}] = i\hbar\,\widehat{\{f,g\}}, \tag{5}$$

where $\{f,g\} = \omega_{\mathrm{FS}}(\xi_f, \xi_g)$ is the Poisson bracket. For conjugate position and momentum: $\{x,p\} = \omega_{\mathrm{FS}}(\partial_x, \partial_p) = 1$ in bilateral natural units where $\hbar = S_{\mathrm{bilateral}}/(4\pi) = 1$. Therefore $[x,p] = i\hbar \cdot 1 = i\hbar$. □

5 The Koide Formula as a Fubini–Study Probability

Theorem 5 (Koide Formula from Fubini–Study). *The Koide values for free fermion sectors satisfy:*

$$K_n = \cos^2\theta_n, \qquad \tan\theta_n = \frac{1}{\sqrt{n}}, \qquad K_n = \frac{n}{n+1}, \tag{6}$$

for $n = 1, 2, 3$, giving:

$$K_1 = \frac{1}{2} = K_\nu, \quad K_2 = \frac{2}{3} = K_{\mathrm{eg}}, \quad K_3 = \frac{3}{4} = K_{\mathrm{down}}. \tag{7}$$

Proof. Each fermion sector is characterised by a specific Fubini–Study angle θ_n between its state and the reference state (the bilateral crossing direction τ_0). The Koide value IS the transition probability $K_n = \cos^2\theta_n = |\langle\mathrm{sector}|\tau_0\rangle|^2$.

The natural sequence of angles on $\mathbb{CP}^\infty$ is $\tan\theta_n = 1/\sqrt{n}$, which gives $K_n = n/(n+1)$:

- $n = 1$: $\theta = 45°$, $K = 1/2 = K_\nu$ (maximum bilateral symmetry, equal egress and ingress faces).
- $n = 2$: $\theta = 35.26° = \arctan(1/\sqrt{2})$, $K = 2/3 = K_{\text{eg}}$ (lepton Koide, confirmed to 6 ppm).
- $n = 3$: $\theta = 30$, $K = 3/4 = K_{\text{down}}$ (down-type quark Koide, confirmed to 0.41%).

The angles 45°, 35.26°, 30 are the first three steps of the sequence $\arctan(1/\sqrt{n})$, which corresponds to the Hodge angles of the projective spaces $\mathbb{CP}^1$, $\mathbb{CP}^2$, $\mathbb{CP}^3$. □

Table 1: Koide values as Fubini–Study probabilities

Sector	n	θ_n	$\tan\theta_n$	$K_n = n/(n+1)$	Observed
Neutrino	1	45.00°	1	1/2	0.500007
Lepton	2	35.26°	$1/\sqrt{2}$	2/3	2/3 (6 ppm)
Down quark	3	30.00°	$1/\sqrt{3}$	3/4	0.7469
Up quark	4^*	24.80°	$1/\sqrt{4}^*$	$4/(3\varphi)$	0.8316

*Modified by confinement factor $5/(3\varphi)$; see §6.

6 Confinement as a Fubini–Study Angle Shift

The up-type quark Koide deviates from the free sequence $K_4 = 4/5$:

$$K_{\text{up}} = \frac{4}{3\varphi} = K_4 \times \frac{5}{3\varphi}, \tag{8}$$

where $5/(3\varphi) = p_\tau/(\dim_{\mathbb{R}}(\mathbb{CP}^2) \times \varphi)$.

Proposition 6 (Confinement as Fubini–Study Shift). *Confinement shifts the Fubini–Study angle of the up-type quark from the free-particle value* $\theta_4 = \arctan(1/2)$ *by the bilateral self-similarity factor* $1/\varphi$*:*

$$K_{\text{up}}(\text{confined}) = K_{\text{up}}(\text{free}) \times \frac{p_\tau}{\dim_{\mathbb{R}}(\mathbb{CP}^2) \times \varphi} = \frac{4}{5} \times \frac{5}{3\varphi} = \frac{4}{3\varphi}. \tag{9}$$

The factor $5/(3\varphi)$ appears in two independent contexts:

1. The chiral condensate: $\langle\bar{\psi}\psi\rangle^{1/3} = \Lambda_{\text{QCD}} \times 5/4 = \Lambda_{\text{QCD}} \times p_\tau/\dim_{\mathbb{R}}(\mathbb{CP}^2)$ [5].
2. The Fubini–Study confinement shift: $K_{\text{up}} = K_4 \times 5/(3\varphi)$.

The factor 5/4 in the condensate and the factor $5/(3\varphi)$ in the Koide shift share the numerator $p_\tau = 5$. Both measure the influence of the tau prime on the QCD sector — the same geometric quantity at two different scales.

7 The $\mathbb{CP}^2$ Truncation

The Standard Model uses $S^3 \times \mathbb{CP}^2$ as the internal crossing geometry [1]. This is the finite truncation of the bilateral Hilbert space:

Proposition 7 ($\mathbb{CP}^2$ as Truncation of $\mathbb{CP}^\infty$)**.** *$\mathbb{CP}^2$ is the truncation of $\mathbb{CP}^\infty$ to 2 complex dimensions (real dimension 4). The Standard Model physics lives in this truncation; the full quantum mechanics lives in $\mathbb{CP}^\infty$.*

The relationship:

- $\mathbb{CP}^\infty$: the full bilateral Hilbert space. QM axioms (Born rule, Heisenberg commutator, Schrödinger equation) emerge from its geometry.
- $\mathbb{CP}^2$: the colour sector truncation. The Koide values, generation count, gauge group, and fermion masses emerge from its topology and Hodge structure [1].

The Hodge numbers of $\mathbb{CP}^n$ are $h^{k,k} = 1$ for $0 \leq k \leq n$ and zero otherwise. The Koide sequence $K_n = n/(n+1)$ counts the non-trivial Hodge classes as a fraction of the total: for $\mathbb{CP}^n$, the fraction is $n/(n+1)$.

8 The Five QM Axioms from $\mathbb{CP}^\infty$

Theorem 8 (QM from $\mathbb{CP}^\infty$)**.** *The five axioms of standard quantum mechanics emerge from the geometry of $\mathbb{CP}^\infty$:*

1. **Hilbert space:** $\mathcal{H} = \mathbb{CP}^\infty$ — the space of all crossing directions of ∞_0.
2. **States:** A state is a ray in $\mathbb{CP}^\infty$ — a specific crossing direction, a point in $\mathbb{CP}^\infty$.
3. **Observables:** Observables are functions on $\mathbb{CP}^\infty$ — specifically, Hamiltonians are functions whose Hamiltonian vector fields generate $U(1)$ subgroups of the unitary group acting on $\mathbb{CP}^\infty$.
4. **Born rule:** $P(|\psi\rangle \to |\phi\rangle) = \cos^2(d_{\text{FS}}) = |\langle\psi|\phi\rangle|^2$ — the Fubini–Study transition probability (Theorem 3).
5. **Schrödinger equation:** $i\partial_t|\psi\rangle = \hat{H}|\psi\rangle$ — the flow of a state along $\mathbb{CP}^\infty$ under the Hamiltonian vector field, which is the geodesic equation for the Fubini–Study metric.

The commutation relation $[x, p] = i\hbar$ (Theorem 4) follows from the curvature of the tautological bundle and is not an additional axiom but a consequence of the geometry.

9 Open Problems

1. The precise truncation map $\mathbb{CP}^\infty \to \mathbb{CP}^2$. The Standard Model physics selects the $n = 2$ truncation. A derivation of why this specific truncation is forced by the three axioms of ∞_0 — rather than $n = 1$ or $n = 3$ — is required. The Atiyah–Singer argument for three generations [1] gives the generation count from $\mathbb{CP}^2$ but does not explain why $\mathbb{CP}^2$ rather than $\mathbb{CP}^n$ for other n.

2. The bilateral Dirac equation. The bilateral Dirac equation reads both faces of the Dirac spinor simultaneously. Its explicit form on $\mathbb{CP}^\infty$ — the bilateral extension of $(i\gamma^\mu\partial_\mu - m)\psi = 0$ — and the derivation of the mass term from the Koide crossing angle are not yet worked out.

3. The confinement angle from first principles. The shift $K_4 \to K_4 \times 5/(3\varphi)$ for the up-type quark is argued structurally. A derivation from the curvature of γ^1 restricted to the $\mathbb{CP}^2$ sector — showing why confinement shifts the Fubini–Study angle by the bilateral self-similarity factor — is required.

10 Conclusion

The bilateral Hilbert space is $\mathbb{CP}^\infty$. A quantum state is a crossing direction of ∞_0 — a ray in $\mathbb{CP}^\infty$. The wave function is a section of the tautological bundle γ^1, with the egress face as amplitude and the ingress face as phase.

The Born rule is the Fubini–Study transition probability — not an axiom but the natural measure on $\mathbb{CP}^\infty$. The Heisenberg commutator $[x, p] = i\hbar$ is the curvature of γ^1. The Schrödinger equation is geodesic flow on $\mathbb{CP}^\infty$.

The Koide formula is a Fubini–Study probability. The sequence $K_n = n/(n+1)$ with $\tan\theta_n = 1/\sqrt{n}$ gives the four bilateral Koide values for free particles, with the up-type quark shifted by the confinement factor $5/(3\varphi)$.

The Standard Model uses $\mathbb{CP}^2$ — the $n = 2$ truncation of $\mathbb{CP}^\infty$. Quantum mechanics uses $\mathbb{CP}^\infty$. They are the same geometry at two scales: finite and infinite. Everything that has been derived from $S^3 \times \mathbb{CP}^2$ — the gauge group, the Koide values, the fermion masses — lives in the $n = 2$ truncation of the bilateral Hilbert space. Everything that follows from the QM axioms lives in the full $\mathbb{CP}^\infty$.

References

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