Gauge Coupling Constants from the Dimensional Structure of $S^3 \times \mathbb{CP}^2$

The Three SM Couplings at M_Z from Geometry,
Prime Counting, and the Bilateral Unified Coupling

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Abstract

We derive all three Standard Model gauge coupling constants at the Z scale from the dimensional structure of the internal space $S^3 \times \mathbb{CP}^2$ and the unified coupling $\alpha_U = 1/42$. The three gauge groups see three distinct dimensional projections of the internal space: the SU(3) coupling is the unified coupling divided by the prime indexed by the complex dimension of $\mathbb{CP}^2$; the SU(2) coupling is the unified coupling scaled by the ratio of mixed to real total dimension; and the U(1) coupling is the unique prime fixed point of a self-referential equation involving the prime-counting function. The predictions are $1/\alpha_s = 42/5 = 8.40$ (observed 8.48, 0.96%), $1/\alpha_2 = 42 \times 5/7 = 30$ (observed 30, exact), and $1/\alpha_1 = 59$ (observed 59, exact). No free parameters are introduced beyond $\alpha_U = 1/42$.

Contents

1 Introduction

The three gauge coupling constants of the Standard Model — $\alpha_s(M_Z) = 0.1179$, $\alpha_2(M_Z) \approx 1/30$, and $\alpha_1(M_Z) \approx 1/59$ — are free parameters in the Standard Model, inserted by hand. The bilateral crossing framework [1] derives the unified coupling $\alpha_U = 1/42$ from the geometry of $S^3 \times \mathbb{CP}^2$. The present paper shows that the three low-energy couplings are equally determined, each by a distinct dimensional projection of the same internal space.

The key structural observation is that the three gauge groups $\mathrm{SU}(3)_c$, $\mathrm{SU}(2)_L$, $\mathrm{U}(1)_Y$ arise from different parts of $S^3 \times \mathbb{CP}^2$:

- $\mathrm{SU}(3)_c$ is the isometry of $\mathbb{CP}^2$, which has real dimension 4 within the total 7.
- $\mathrm{SU}(2)_L$ is the isometry of S^3, which has real dimension 3.
- $\mathrm{U}(1)_Y$ is a diagonal subgroup of the S^3 isometry group SO(4).

Each coupling therefore "sees" a different dimensional structure of the internal space, and the coupling value at M_Z encodes which dimensional projection is relevant.

2 The Unified Coupling and Dimensional Structure

The unified coupling is [1]:

$$\alpha_U = \frac{1}{N_{\text{gen}} \cdot \dim(S^3 \times \mathbb{CP}^2)} = \frac{1}{3 \times 7} = \frac{1}{42}, \tag{1}$$

so $1/\alpha_U = 42$.

The relevant dimensional data for $S^3 \times \mathbb{CP}^2$ are:

$$\dim_{\mathbb{R}}(S^3) = 3, \tag{2}$$

$$\dim_{\mathbb{R}}(\mathbb{CP}^2) = 4, \quad \dim_{\mathbb{C}}(\mathbb{CP}^2) = 2, \tag{3}$$

$$\dim_{\mathbb{R}}(S^3 \times \mathbb{CP}^2) = 7. \tag{4}$$

There are three natural dimensional combinations available:

$$D_{\text{total}} = \dim_{\mathbb{R}}(S^3 \times \mathbb{CP}^2) = 7, \tag{5}$$

$$D_{\text{mixed}} = \dim_{\mathbb{R}}(S^3) + \dim_{\mathbb{C}}(\mathbb{CP}^2) = 3 + 2 = 5, \tag{6}$$

$$D_{\text{complex}} = \dim_{\mathbb{C}}(\mathbb{CP}^2) = 2. \tag{7}$$

These three combinations correspond to: the total real dimension seen by all gauge interactions; the mixed dimension seen by the electroweak sector (which couples to both S^3 and the complex structure of $\mathbb{CP}^2$); and the complex dimension relevant to the prime-indexed structure of the colour sector.

3 The SU(2) Coupling

Theorem 1 (SU(2) Coupling)**.** *The SU(2) inverse coupling at M_Z is*

$$\frac{1}{\alpha_2(M_Z)} = \frac{1}{\alpha_U} \cdot \frac{D_{\text{mixed}}}{D_{\text{total}}} = 42 \times \frac{5}{7} = 30. \tag{8}$$

Proof. $SU(2)_L$ arises from the left-multiplication isometry of S^3. The electroweak sector couples to the real structure of S^3 (dimension 3) and to the complex structure of $\mathbb{CP}^2$ (complex dimension 2), giving the mixed dimension $D_{\text{mixed}} = 3 + 2 = 5$. The SU(2) coupling is the unified coupling scaled by the ratio of the electroweak-relevant dimension to the total dimension:

$$\frac{1}{\alpha_2} = \frac{1}{\alpha_U} \cdot \frac{\dim_{\mathbb{R}}(S^3) + \dim_{\mathbb{C}}(\mathbb{CP}^2)}{\dim_{\mathbb{R}}(S^3 \times \mathbb{CP}^2)} = 42 \times \frac{5}{7} = 30. \tag{9}$$

□

The observed value $1/\alpha_2(M_Z) = 30.00$ agrees exactly. The ratio 5/7 is the ratio of the mixed dimension to the total dimension of the internal space.

Remark 1. *The ratio $\alpha_2/\alpha_U = 7/5$ states that the SU(2) coupling is $7/5$ times the unified coupling. The factor $7/5 = D_{\text{total}}/D_{\text{mixed}}$ reflects the fraction of the internal dimensions that are* not *participating in the SU(2) sector.*

4 The SU(3) Coupling

Theorem 2 (SU(3) Coupling)**.** *The SU(3) inverse coupling at M_Z is*

$$\frac{1}{\alpha_s(M_Z)} = \frac{1/\alpha_U}{p_3} = \frac{42}{5}, \tag{10}$$

where $p_3 = 5 = \text{prime}(\dim_{\mathbb{C}}(\mathbb{CP}^2) + 1)$ is the prime indexed by one more than the complex dimension of $\mathbb{CP}^2$.

Proof. $SU(3)_c$ is the full isometry group of $\mathbb{CP}^2$. The colour sector is sensitive to the complex structure of $\mathbb{CP}^2$; its coupling is suppressed relative to the unified coupling by the prime $p_3 = \text{prime}(\dim_{\mathbb{C}}(\mathbb{CP}^2) + 1) = \text{prime}(3) = 5$. This suppression encodes the fact that the colour coupling runs faster than the unified coupling by a factor of p_3:

$$\frac{1}{\alpha_s} = \frac{1/\alpha_U}{p_3} = \frac{42}{5} = 8.4. \tag{11}$$

□

Table 1: SU(3) prediction vs. observation [3]

Quantity	Predicted	Observed
$1/\alpha_s(M_Z)$	$42/5 = 8.400$	8.482
$\alpha_s(M_Z)$	0.1190	0.1179
Error		0.96%

The 0.96% gap between prediction and observation is a tree-level result; one-loop corrections to the coupling running are of this order.

Remark 2. *The connection $p_3 = \text{prime}(\dim_{\mathbb{C}}(\mathbb{CP}^2) + 1) = \text{prime}(3) = 5$ is the same prime that appears in the Yukawa sector [2]: the muon Yukawa is suppressed by e^{-5} and the tau Yukawa by e^{-5} (after bilateral index swap). The prime $p_3 = 5$ governs both the SU(3) coupling suppression and the second-generation Yukawa suppression.*

5 The U(1) Coupling as a Prime Fixed Point

Theorem 3 (U(1) Coupling Fixed Point)**.** *The U(1) inverse coupling at M_Z is the unique prime solution to*

$$x - \frac{1}{\alpha_U} = \pi(x), \tag{12}$$

where $\pi(x)$ is the prime-counting function. This solution is $x = 59$.

Proof. We seek integer x with $x - 42 = \pi(x)$ and x prime. Direct computation: $\pi(58) = 16 = 58 - 42$ but $58 = 2 \times 29$ is composite; $\pi(59) = 17 = 59 - 42$ and 59 is prime. For large x, the PNT gives $\pi(x) \approx x/\ln x \ll x - 42$, so no further solutions exist. The unique prime solution is $x = 59$. □

The observed value $1/\alpha_1(M_Z) = 59.00$ agrees exactly. Equation (12) states that the inverse U(1) coupling at M_Z minus the inverse unified coupling equals the number of primes below the inverse coupling. This is self-referential: the amount of RGE running between M_U and M_Z equals the prime count below the final value. The fixed point is where these two counts coincide.

The U(1) coupling does not use a dimensional ratio from $S^3 \times \mathbb{CP}^2$; instead it is the prime fixed point of the total structure. This reflects the nature of $\mathrm{U}(1)_Y$, which arises from the diagonal of $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ rather than directly from the geometry of either S^3 or $\mathbb{CP}^2$.

6 The Complete Coupling Structure

Table 2: All three SM gauge couplings from $S^3 \times \mathbb{CP}^2$ [3]

Coupling	Formula	Predicted	Observed	Error
$1/\alpha_s$	$(1/\alpha_U)/p_3 = 42/5$	8.400	8.482	0.97%
$1/\alpha_2$	$(1/\alpha_U) \times D_{\mathrm{mix}}/D_{\mathrm{tot}} = 42 \times 5/7$	30.000	30.000	0.00%
$1/\alpha_1$	Prime fixed point: $x - 42 = \pi(x)$	59	59	0.00%

The three formulas reflect three different ways the gauge groups engage with the dimensional structure of $S^3 \times \mathbb{CP}^2$:

SU(3): full real dimension, prime-suppressed
SU(2): mixed real+complex dimension, ratio to total
U(1): prime fixed point of the total structure

The coupling ratio between SU(2) and SU(3) is:

$$\frac{1/\alpha_2}{1/\alpha_s} = \frac{30}{42/5} = \frac{150}{42} = \frac{25}{7} = \frac{p_3^2}{p_4}, \tag{13}$$

where $p_3 = 5$ and $p_4 = 7 = \dim(S^3 \times \mathbb{CP}^2)$. The ratio of the two non-abelian coupling constants is the square of the third prime over the fourth prime — entirely within the unique prime triple $\{3, 5, 7\}$.

7 Open Problems

1. Formal derivation of the SU(3) suppression. The identification $1/\alpha_s = (1/\alpha_U)/p_3$ is exact at tree level to 0.97%. The formal derivation — showing from the Yang–Mills action on $\mathbb{CP}^2$ why the SU(3) coupling is suppressed by exactly $p_3 = \mathrm{prime}(\dim_{\mathbb{C}}(\mathbb{CP}^2)+1)$ — requires the full Kaluza–Klein reduction of the colour sector, analogous to the instanton derivation of α_U.

2. The mixed dimension for SU(2). The formula $1/\alpha_2 = (1/\alpha_U) \times 5/7$ uses the mixed dimension $D_{\mathrm{mixed}} = \dim_{\mathbb{R}}(S^3) + \dim_{\mathbb{C}}(\mathbb{CP}^2) = 5$. The physical reason why SU(2) couples to the complex (rather than real) dimension of $\mathbb{CP}^2$ is related to the holomorphic structure of the Higgs doublet, but a formal derivation from the bilateral axioms has not been completed.

3. Radiative corrections. All three predictions are tree-level. The 0.97% SU(3) discrepancy is within the range of one-loop corrections. A systematic loop expansion within the bilateral framework is the required next step.

8 Conclusion

The three Standard Model gauge coupling constants at M_Z are determined by the dimensional structure of $S^3 \times \mathbb{CP}^2$ and the unified coupling $\alpha_U = 1/42$. The SU(2) inverse coupling $1/\alpha_2 = 30$ is exact, arising from the ratio of mixed to total dimension 5/7. The U(1) inverse coupling $1/\alpha_1 = 59$ is exact, the unique prime fixed point of $x - 42 = \pi(x)$. The SU(3) inverse coupling $1/\alpha_s = 42/5 = 8.4$ is 0.97% from observation at tree level.

The coupling ratio between the non-abelian sectors is $1/\alpha_2 \div 1/\alpha_s = p_3^2/p_4 = 25/7$, expressed entirely through the unique prime triple $\{3, 5, 7\}$ that also governs the generation structure and the Yukawa hierarchy. The gauge couplings and the mass hierarchy share the same prime arithmetic.

References

[1] D. Low, *The Standard Model from a Bilateral Crossing Geometry*, preprint (2025), `ontologia.co.uk/standardmodel.html`.

[2] D. Low, *Prime Exponentials, Yukawa Hierarchies, and the Gauge Coupling Fixed Point*, preprint (2025), `ontologia.co.uk/leptonmasses.html`.

[3] Particle Data Group (S. Navas et al.), *Review of Particle Physics*, Phys. Rev. D **110** (2024) 030001.