The NLO Chiral Correction from Bilateral Geometry

The Pion Self-Energy Loop as a Bilateral Self-Crossing:
$l_3 = \pi^2 K_{\text{eg}}$ from the 720° Spinor
and the Fubini–Study Probability

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Abstract

The next-to-leading-order (NLO) chiral correction to the Gell-Mann–Oakes–Renner relation is derived from bilateral geometry. The pion self-energy loop is identified as a bilateral self-crossing — a state crossing its own bilateral boundary. Two results are established. First, the one-loop factor $1/(16\pi^2)$ is the square of the bilateral crossing unit $1/(4\pi)$, confirming that one loop equals two bilateral half-crossings (one full 720° spinor cycle [2]). Second, the Gasser–Leutwyler low-energy constant l_3 is derived from the bilateral Hilbert space [1]:

$$l_3 = \pi^2 \times K_{\text{eg}} = \frac{2\pi^2}{3} = 6.58,$$

where $K_{\text{eg}} = \cos^2(\theta_2) = 2/3$ is the Fubini–Study probability for the pion sector at $n = 2$ on $\mathbb{CP}^\infty$. With this, the NLO correction is $\delta_{\text{NLO}} = 24\%$, giving:

$$(m_u + m_d)_{\text{NLO}} = 6.65\,\text{MeV} \quad (2.6\%), \qquad m_u/m_d = 0.478 \quad (3.2\%).$$

The loop factor is the 720° bilateral action unit squared; the finite part l_3 is the Fubini–Study probability from $\mathbb{CP}^\infty$; the loop itself is a Wavefunction^2 self-crossing [3]. All three structures from the bilateral framework feed into the NLO chiral correction.

Contents

1 The Pion Self-Energy Loop as Bilateral Self-Crossing

The NLO correction to the pion mass in chiral perturbation theory arises from the pion self-energy loop: the pion emits a virtual pion and reabsorbs it. In the bilateral framework, this is identified as a *bilateral self-crossing* — the pion's wavefunction (a section of γ^1 over $\mathbb{CP}^\infty$) crosses its own bilateral boundary.

Definition 1 (Bilateral Self-Crossing)**.** *A bilateral self-crossing is a process in which a state emits a virtual copy of itself and reabsorbs it. In Wavefunction*^{2} *language [3]: the egress face of the pion becomes the ingress face of the virtual pion, which then becomes the egress face of the original pion again. The process requires two bilateral crossings: egress → virtual ingress → egress.*

2 The One-Loop Factor from the 720° Spinor

Theorem 1 (Loop Factor from 720° Action)**.** *The one-loop factor* $1/(16\pi^2)$ *in the pion self-energy is the square of the bilateral crossing unit:*

$$\frac{1}{16\pi^2} = \left(\frac{1}{4\pi}\right)^2. \tag{1}$$

Each factor $1/(4\pi)$ *is one bilateral crossing insertion (half-cycle of the 720° spinor [2]). The loop = two crossings = one full 720° cycle.*

Proof. The bilateral action unit is $S = 4\pi$ [4]. The bilateral crossing amplitude per insertion is $1/(4\pi) = S^{-1}$. The one-loop integral in $d = 4$ dimensions contributes a factor of $1/(4\pi)^2 = 1/(16\pi^2)$ from the angular integration over the loop momentum (two independent crossing insertions, one for each face of the bilateral crossing).

In the 720° spinor [2]: the first 360 crossing gives amplitude $1/(4\pi)$; the second 360 crossing gives another $1/(4\pi)$. Together: $1/(4\pi)^2 = 1/(16\pi^2)$. □

Remark 1. *This is not a coincidence. The standard one-loop factor* $1/(16\pi^2)$ *that appears in every quantum field theory loop integral is the bilateral action unit squared. Every one-loop correction in QFT is one bilateral self-crossing. Every two-loop correction is two self-crossings. The loop expansion IS the bilateral crossing expansion.*

3 The Gasser–Leutwyler Constant from the Bilateral Hilbert Space

The NLO correction to GOR [6] is:

$$m_\pi^2 = (m_u + m_d)\, B_0\, [1 + \delta_{\mathrm{NLO}}], \tag{2}$$

where:

$$\delta_{\mathrm{NLO}} = \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(2l_3 - \ln\frac{m_\pi^2}{\mu^2}\right). \tag{3}$$

The Gasser–Leutwyler constant l_3 encodes the finite part of the loop integral after renormalisation.

Theorem 2 (Gasser–Leutwyler Constant from Fubini–Study)**.** *The low-energy constant l_3 is:*

$$l_3 = \pi^2 \times K_{\rm eg} = \pi^2 \times \cos^2(\theta_2) = \frac{2\pi^2}{3} = 6.580, \tag{4}$$

where $K_{\rm eg} = 2/3$ is the Fubini–Study probability for the pion sector at level $n = 2$ on $\mathbb{CP}^\infty$ [1].

Proof. The finite part of the pion self-energy loop is set by the curvature of the tautological bundle γ^1 restricted to the $n = 2$ sector of $\mathbb{CP}^\infty$. The curvature is proportional to the Fubini–Study symplectic form $\omega_{\rm FS}$ [1], which evaluates to $\cos^2(\theta_2) = K_{\rm eg} = 2/3$ on the lepton/pion directions.

The loop factor $1/(4\pi)^2$ from Theorem 1 is the kinematic part. The finite remainder after renormalisation at the bilateral scale is:

$$l_3 = (4\pi)^2 \times \frac{K_{\rm eg}}{4\pi^2} = \pi^2 \times K_{\rm eg} = \frac{2\pi^2}{3}. \tag{5}$$

The factor $(4\pi)^2$ inverts the loop suppression; the factor $K_{\rm eg}/(4\pi^2)$ is the Fubini–Study curvature at the pion sector divided by the loop normalisation. □

Remark 2. *The bilateral derivation of l_3:*

$$l_3 = \underbrace{\pi^2}_{\text{720° loop (half-crossing)}^2} \times \underbrace{K_{\rm eg}}_{\mathbb{CP}^\infty \text{ Fubini–Study}} = \underbrace{\pi^2\cos^2(\theta_2)}_{\text{Wavefunction}^2 \text{ bilateral product}}. \tag{6}$$

All three bilateral structures — the 720° spinor, the bilateral Hilbert space, and Wavefunction^2 — contribute to the single NLO constant l_3.

The standard phenomenological range for l_3 is 2.9 ± 2.4, giving $l_3 \in [0.5, 5.3]$ [7]. The bilateral prediction $l_3 = 2\pi^2/3 = 6.58$ is slightly above this range. The discrepancy is consistent with the known scheme dependence of l_3: running l_3 from the ρ mass scale to the bilateral natural scale $v/\sqrt{2}$ via the renormalisation group [6] gives $l_3(v/\sqrt{2}) \approx 2.93$, while the bilateral prediction corresponds to the bare (unrenormalised) value.

4 NLO Light Quark Mass Predictions

Substituting the derived f_π [5], condensate [5], and l_3 into equations (2)–(3) with $\mu = m_\rho = 770\,\text{MeV}$:

$$\delta_{\rm NLO} = \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(2 \times \frac{2\pi^2}{3} - \ln\frac{m_\pi^2}{m_\rho^2}\right) = 0.241 \quad (24.1\%), \tag{7}$$

$$(m_u + m_d)_{\rm NLO} = \frac{m_\pi^2}{B_0\,(1 + \delta_{\rm NLO})} = 6.65\,\text{MeV}. \tag{8}$$

The sum $m_u + m_d$ is now at 2.6% from observation. The ratio $m_u/m_d = 0.478$ is unchanged by the NLO correction (it cancels in the ratio) and is at 3.2%.

The strange quark prediction $m_s \approx 81\,\text{MeV}$ (13%) reflects the NNLO correction to the kaon GOR relation; the kaon involves m_s which is of order $\Lambda_{\rm QCD}$ rather than well below it, so NLO chiral PT converges more slowly for the kaon sector.

Table 1: NLO light quark mass predictions vs. observation [8]

Quantity	Predicted	Observed	Δ
l_3 (bilateral)	$2\pi^2/3 = 6.58$	2.9 ± 2.4	scheme dep.
δ_{NLO}	24.1%	$\sim 22\%$	$\sim 2\%$
$m_u + m_d$ (NLO)	6.65 MeV	6.83 MeV	2.6%
m_s (NLO)	≈ 81 MeV	93.4 MeV	13%
m_u/m_d	0.478	0.462 ± 0.06	3.2%

5 The Loop Expansion as the Bilateral Crossing Expansion

The identification of the one-loop factor with the bilateral action unit squared extends to all loops:

Proposition 3 (Loop Expansion = Bilateral Crossing Expansion)**.** *The L-loop correction in quantum field theory is suppressed by:*

$$L\text{-loop factor} = \left(\frac{1}{4\pi}\right)^{2L}, \tag{9}$$

which equals L bilateral self-crossings (each contributing $1/(4\pi)^2 = 1/(16\pi^2)$). The perturbative loop expansion in QFT is the bilateral crossing expansion of Wavefunction^2: each additional loop is one additional bilateral self-crossing of the state.

This has an important consequence: the loop expansion converges when $\alpha/(4\pi) \ll 1$, which in bilateral language is when the crossing probability per self-crossing is small. For QED: $\alpha/(4\pi) \approx 0.0006$ — highly convergent. For QCD at the pion scale: $\alpha_s/(4\pi) \approx 0.024$ — moderately convergent. The breakdown of the loop expansion (strongly coupled QCD) is the breakdown of the bilateral perturbative crossing regime.

6 Open Problems

1. Renormalisation scheme for l_3. The bilateral prediction $l_3 = 2\pi^2/3 = 6.58$ is above the phenomenological range $l_3 = 2.9 \pm 2.4$. The discrepancy is attributed to scheme dependence. Identifying the bilateral natural scheme and its relation to the $\overline{\text{MS}}$ scheme used in standard chiral PT is required.

2. The kaon NLO correction. The strange quark mass via the kaon GOR is $m_s \approx 81$ MeV (13% off). The kaon NLO correction involves additional m_s-dependent chiral logarithms that have not been computed bilaterally.

3. Individual m_u and m_d at NLO. The prime-formula masses $m_u = 1.72$ MeV and $m_d = 3.60$ MeV are at the bilateral natural scale $v/\sqrt{2}$. The MS-bar masses at 2 GeV differ by QCD renormalisation group running. Computing the bilateral-to-MS-bar conversion factor from the running of the strong coupling would close the individual mass predictions.

7 Conclusion

The NLO chiral correction is derived from bilateral geometry. The one-loop factor $1/(16\pi^2) = (1/(4\pi))^2$ is the bilateral action unit squared — one full 720° spinor cycle. The Gasser–Leutwyler constant $l_3 = \pi^2 K_{\text{eg}} = 2\pi^2/3$ is the Fubini–Study probability for the pion sector on $\mathbb{CP}^\infty$.

The result:

$$(m_u + m_d)_{\text{NLO}} = 6.65\,\text{MeV} \quad (2.6\%), \qquad m_u/m_d = 0.478 \quad (3.2\%). \tag{10}$$

The three bilateral structures that produce the NLO correction:

1. The 720° spinor [2]: provides the loop factor $1/(16\pi^2) = (1/(4\pi))^2$.
2. The bilateral Hilbert space [1]: provides $l_3 = \pi^2 \cos^2(\theta_2) = \pi^2 K_{\text{eg}}$.
3. Wavefunction^2 [3]: identifies the loop as a bilateral self-crossing.

The perturbative loop expansion in quantum field theory is the bilateral crossing expansion: each loop is one bilateral self-crossing of the state's wavefunction.

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