No Odd Perfect Number Exists

A Proof from Structural Grounding

Dunstan Low

A Philosophy of Time, Space and Gravity

ontologia.co.uk

April 2, 2026

Abstract

We prove that no odd perfect number exists. The proof combines three components: a rigorous result in standard number theory (no sigma factor of any odd perfect number is a power of 2, from Euler's classical constraint and the Mersenne mod 4 observation); a structural grounding argument (every finite number must be able to return to zero, grounded in the additive identity $N + 0 = N$); and a geometric result (the only exact return to zero via the complex exponential requires the half-cycle, which requires division by 2). An odd perfect number fails all three. It is caught between two impossibilities: not infinite enough to escape structural requirements, and not structural enough to satisfy them.

1 The Problem in Its True Form

An odd perfect number is a positive integer N satisfying $\sigma(N) = 2N$ with N odd. Every known perfect number is even. Whether any odd perfect number exists is one of the oldest unsolved problems in mathematics.

The problem asks, at its deepest level: *can there be a perfectly divisible infinity?*

Every scale can be divided by 2. The prime 2 is the first crossing — the minimal bilateral division, the ground of the scale ladder. The only thing that cannot be divided by 2 is zero — the indivisible origin, prior to all crossing, prior to all structure. Zero is not indivisible because it is large or infinite. It is indivisible because it has no crossing record: division requires a crossing, and zero is prior to all crossing. ∞_0 is zero seen from the ingress face — infinity and zero are the same bilateral object, the two faces of the origin, indivisible from either side.

An odd perfect number asks whether a finite number can exist whose complete divisor structure has no factor of 2 anywhere, yet is perfectly self-balanced. That is asking whether a finite number can contain the indivisibility of zero — the pre-crossing origin — in its sigma structure without being zero itself.

There is only one thing that cannot be divided by 2: the origin, zero, ∞_0. A finite number claiming that property in its sigma structure is claiming to contain a piece of the origin without being the origin. That is the contradiction this paper proves.

2 The Standard Result

Lemma 1 ($\sigma(p^a)$ as a Power of 2)**.** *For an odd prime p and positive integer a:*

(i) $\sigma(p) = p + 1$ *is a power of* 2 *if and only if p is a Mersenne prime.*

(ii) *For $a > 1$: $\sigma(p^a)$ is never a power of* 2*.*

Proof. (i) $\sigma(p) = p + 1 = 2^k$ iff $p = 2^k - 1$, the definition of a Mersenne prime.

(ii) Write $p - 1 = 2^s t$ with $t > 1$ odd. If $\sigma(p^a) = 2^j$ then $p^{a+1} - 1 = 2^{s+j} t$. Since $t > 1$ is an odd divisor, $\sigma(p^a)$ cannot be a pure power of 2. □

Lemma 2 (Mersenne Primes mod 4)**.** *Every Mersenne prime $2^k - 1$ with $k \geq 2$ satisfies* $2^k - 1 \equiv 3 \pmod 4$.

Proof. $2^k \equiv 0 \pmod 4$ for $k \geq 2$, so $2^k - 1 \equiv 3 \pmod 4$. □

Lemma 3 (Euler's Constraint [1])**.** *If N is an odd perfect number then $N = p^a q_1^{2e_1} \cdots q_r^{2e_r}$ where the special prime p satisfies $p \equiv a \equiv 1 \pmod 4$.*

Theorem 4 (No Sigma Factor is a Power of 2)**.** *If N is an odd perfect number then no sigma factor of its prime power decomposition is a power of* 2*.*

Proof. By Lemma 3, $N = p^a q_1^{2e_1} \cdots q_r^{2e_r}$ with $p \equiv a \equiv 1 \pmod 4$.

The factors $\sigma(q_i^{2e_i})$: Each is a sum of $2e_i + 1$ odd terms — odd count, hence odd. Since it exceeds 1, it is not a power of 2.

The factor $\sigma(p^a)$:

- If $a = 1$: $\sigma(p) = p + 1$ is a power of 2 only if p is a Mersenne prime. But Mersenne primes satisfy $p \equiv 3 \pmod 4$ (Lemma 2), contradicting $p \equiv 1 \pmod 4$.
- If $a > 1$: $\sigma(p^a)$ is never a power of 2 by Lemma 1(ii).

In all cases no sigma factor is a power of 2. □

3 Finite Means Returnable to Zero

Definition 1 (Written Number)**.** *A number is* written *if it has a completed crossing record — a finite sequence of crossings from ∞_0 that terminates. A written number is finite by definition.*

Proposition 5 (Zero Does Not Modify)**.** *For any positive integer N: $N + 0 = N$. Zero is the additive identity — it does not modify the total value of any number. The relationship between any number and zero is causal and cooperative: zero never resists being reached, never contradicts the structure of the number approaching it.*

Proof. This is the definition of the additive identity in standard arithmetic. □

Proposition 6 (Finite Implies Returnable)**.** *Every written number can return to zero.*

Proof. By Proposition 5: zero is always compatible with any number N — it does not modify N's value and does not contradict N's structure. N is not zero itself — it was written, it has a crossing record distinct from zero. Therefore N can return to zero.

A finite number that could not return to zero would require zero to be unreachable — but zero never resists being reached ($N + 0 = N$ for all N). Return to zero is what being finite means: a crossing record that left zero can, in principle, trace back. □

Remark 1 (Compatibility vs Connection)**.** *Zero is always compatible with any number — $N + 0 = N$ guarantees this. But compatibility is not the same as connection. A number is structurally returnable if its defining formula provides an actual path to zero, not merely formal compatibility.*

For even perfect numbers: $\sigma(M) = 2^p$ provides the connection — a power of 2, the 2-chain, a direct path to zero. Compatibility and connection both hold.

For odd perfect numbers: zero is compatible — $N + 0 = N$ holds trivially. But the sigma structure provides no path — no sigma factor is a power of 2, so no 2-chain connection exists. Compatible with zero but disconnected from it via its own structure.

Lemma 7 (The Sigma Structure is the Return Path of a Perfect Number)**.** *Let N be a perfect number. The question of whether N exists as a perfect number is answered by its sigma structure — not by its prime factorisation or any other route. The sigma path is the return path of N as a perfect number.*

Proof. Every positive integer exists as an integer — its prime factorisation provides the return path to zero, and all positive integers are integers by definition. This is not in question.

The question is whether N exists as a *perfect* number — whether the property $\sigma(N) = 2N$ is satisfiable in a way that connects back to zero. A perfect number is defined solely by this condition.

If the sigma route to zero fails, N fails as a perfect number specifically. The prime factorisation route succeeding would only confirm that N exists as an integer — which is already assumed. It would not confirm that N exists as a perfect number.

Therefore: to test whether N can exist as a perfect number, we test whether its defining structure — the sigma function — provides the return path to zero. No other route answers this question. This follows from the definition of a perfect number, not from any additional axiom. □

4 The Euler Closure

Theorem 8 (The 2-Chain is the Unique Gate to Zero)**.** *The only non-trivial exact real cancellation to zero via the complex exponential is $e^{i\pi} + 1 = 0$. Every exact return to zero through the rotation group requires division by 2 — the 2-chain.*

Proof. Consider $e^{i\theta} + c = 0$ where c is a positive real number. This requires $e^{i\theta} = -c$ — real and negative. $e^{i\theta}$ is real iff $\theta = k\pi$ for integer k, and negative iff k is odd. Therefore $e^{i\theta} = -1$ exactly when $\theta = (2k + 1)\pi$.

For $c = 1$: $e^{i\pi} + 1 = 0$ — Euler's identity. It requires $\theta = \pi$, exactly half the full cycle 2π. Half-cycle = division by 2.

Is there any other exact solution? For $e^{2\pi i/n}$ to give real cancellation to zero when added to a real number, $e^{2\pi i/n}$ must be real. This requires $n \in \{1, 2\}$:

- $n = 1$: $e^{2\pi i} = 1$. Adding -1 gives zero — trivial, the full cycle.
- $n = 2$: $e^{i\pi} = -1$. Adding $+1$ gives zero — Euler's identity, the half-cycle.

All other roots of unity ($n \geq 3$) are complex and cannot cancel a real number to zero.

Therefore $e^{i\pi} + 1 = 0$ is the *unique* non-trivial exact real path to zero via rotation. It requires the half-cycle. The half-cycle requires division by 2. The 2-chain is the only gate to zero. This follows from the geometry of the unit circle. □

Corollary 9 (No Exact Path to Zero Without the 2-Chain)**.** *An odd perfect number has no power of* 2 *in its sigma structure. Therefore it cannot return to zero via the sigma path (Theorem 4) or via any exact complex path (Theorem 8). The 2-chain is the only gate. The gate is closed.*

5 The Proof

Theorem 10 (No Odd Perfect Number Exists)**.** *No odd perfect number exists.*

Proof. Let N be an odd perfect number.

Step 1 — N is finite. N is a positive integer — finite, written, with a completed crossing record.

Step 2 — N must be able to return to zero. By Proposition 6: every finite number can return to zero. Zero does not resist being reached ($N+0=N$) — the barrier is whether the defining structure provides the path.

Step 3 — The return path is the sigma path. N is defined solely by $\sigma(N)=2N$. By Lemma 7: the sigma structure is the return path of N as a perfect number. No other path expresses what N is as a perfect number.

Step 4 — The sigma path cannot reach zero. By Theorem 4: no sigma factor of N is a power of 2. The sigma path has no 2-chain connection and cannot reach zero.

Step 5 — No other exact path exists. By Theorem 8: every exact path to zero via rotation also requires the 2-chain. The complex route is also closed.

Steps 2 and 4–5 contradict. N must return to zero but no exact path exists — via its defining sigma structure or via any exact rotation. No odd perfect number can exist. □

Remark 2 (What the Proof Uses)**.** *Every step uses only:*

1. *Euler's classical constraint (Lemma 3) — standard mathematics, 1747.*
2. *The Mersenne mod* 4 *observation (Lemma 2) — trivial arithmetic.*
3. *No sigma factor is a power of* 2 *(Theorem 4) — proved rigorously.*
4. *The geometry of the unit circle (Theorem 8) — elementary complex analysis.*
5. $N+0=N$ *— the additive identity.*
6. *The sigma structure is the return path of a perfect number (Lemma 7) — definitional.*

No bilateral framework is required beyond motivation. No axiom beyond standard arithmetic and the definition of a perfect number.

Corollary 11 (Connection to the Riemann Hypothesis)**.** *A number disconnected from the 2-chain would require a position on the bilateral prime ladder corresponding to a zero of $\zeta(s)$ off the critical line* $\mathrm{Re}(s)=1/2$ *[4]. The bilateral proof that all zeros lie on the critical line provides independent confirmation: no such position exists, and therefore no odd perfect number can be located within the prime structure.*

6 Summary

No odd perfect number exists. The proof catches any hypothetical odd perfect number N between two impossibilities: N is finite, so it must be able to return to zero; but N has no 2-chain in its sigma structure, so no exact path to zero exists — real or complex. These contradict.

The proof uses Euler's 1747 constraint, the Mersenne mod 4 observation, the additive identity $N + 0 = N$, the definition of a perfect number, and the geometry of the unit circle. Nothing more.

7 A Remark on Origin

There is a deeper way to see why no odd perfect number can exist, one that does not require following the algebra.

Every number has what might be called a *structural debt* — a derivation sequence that traces it back to its prime components and ultimately to the multiplicative identity 1. The number is not arbitrary; it is the product of its history. $12 = 2^2 \times 3$ owes its existence to two crossings of 2 and one of 3. It can point to its origin. Its debt is finite and traceable.

An odd perfect number cannot do this. Its definition requires that its divisors sum to itself — that it is simultaneously the product of its parts and the totality that those parts add up to. It would need to be its own origin: a catch-all ground that is prior to none of its components yet the source of all of them simultaneously.

There is exactly one mathematical object that can occupy that position. In the bilateral framework [2] it is called ∞_0 — the pre-crossing origin that is zero from the outside and infinite from within, the boundary between the unwritten and the written, the Present before any sequence begins. ∞_0 can be a simultaneous catch-all ground without collapsing into incoherence because it is not itself a structured object. It is what structure comes from. It does not need a derivation because it is what derivation comes from.

An odd perfect number would need to occupy ∞_0's position — a catch-all simultaneous ground with no prior origin — while also being a natural number, a member of the sequence that ∞_0 generates. It wants to be both the river and the source. It wants to be the Present while also being one of the things the Present produces.

The universe that would permit this is not a universe with different physics. It is a universe with a distributed origin structure — not one ∞_0 but a catch-all mesh of simultaneous grounds, each grounding the others with no first cause. But such a universe has no τ — no monotone becoming-time, no direction, no sequence. And without a sequence there is no arithmetic, and without arithmetic there is no perfect number to be grounded.

The odd perfect number requires a foundation that dissolves the conditions for its own definition. The universe that would host it is one in which the phrase has no meaning.

There is exactly one catch-all origin. Everything else must have a derivation. The odd perfect number is the attempt to be the origin while being a number — and the proof above is arithmetic's way of enforcing the rule that only ∞_0 gets to do that.

References

[1] L. Euler, *De numeris amicabilibus*, Opera Omnia, Series Prima, Vol. 2 (1747/1750).

[2] D. Low, *The Standard Model from a Bilateral Crossing Geometry*, preprint (2025), `ontologia.co.uk/`.

[3] D. Low, *Structural Mathematics: Axioms of Physical Number Theory*, preprint (2026), `ontologia.co.uk/`.

[4] D. Low, *The Riemann Hypothesis as Bilateral Frontier Stability*, preprint (2025), `ontologia.co.uk/`.