Prime Exponentials, Yukawa Hierarchies, and the Gauge Coupling Fixed Point

Mass Hierarchies and Coupling Constants from the Prime Number Theorem
and the Bilateral Crossing Geometry

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Abstract

We show that the lepton mass hierarchy, the Yukawa anomalous dimension, and the U(1) gauge coupling at the Z scale are all determined by the prime number structure of the bilateral crossing geometry. The Yukawa coupling of generation k is $Y_k = Y_0 \, e^{-p_k}$, where p_k is the prime labelling generation k in the unique prime triple $\{3, 5, 7\}$. The prime is the integrated one-loop RGE flow from the unification scale to the generation threshold, with anomalous dimension exactly 1 in bilateral natural units. The mass ratio between generations is therefore $e^{\Delta p}$ where Δp is the prime gap — a direct expression of the prime number theorem. Separately, the U(1) inverse coupling at the Z scale, $1/\alpha_1(M_Z) = 59$, is the unique prime satisfying the self-referential equation $x - 1/\alpha_U = \pi(x)$, where $\pi(x)$ is the prime-counting function and $1/\alpha_U = 42$ is the bilateral unified coupling. No free parameters are introduced.

Contents

1 Introduction

The Standard Model contains nineteen free parameters, including six Yukawa couplings whose values span five orders of magnitude. The hierarchy problem asks why the electroweak scale is so far below the Planck scale. Both questions are conventionally unanswered: the parameters are inserted by hand, and the hierarchy is "explained" only by fine-tuning or new physics.

The bilateral crossing framework [1] derives the gauge group, generation count, Koide ratio, and unified coupling $\alpha_U = 1/42$ from three axioms and the geometry of $S^3 \times \mathbb{CP}^2$. The present paper extends this to the mass hierarchy. The central observation is:

1. The Bohr–Sommerfeld eigenvalues of the egress face are $y_n = p_{n+1}/2$ where $\{p_2, p_3, p_4\} = \{3, 5, 7\}$ is the *unique prime triple* — the only three consecutive odd numbers that are all prime.
2. The Yukawa couplings are prime exponentials: $Y_k = Y_0\, e^{-p_k}$. The prime p_k is the integrated RGE flow from the unification scale to the generation threshold.
3. The U(1) coupling at M_Z is the unique prime fixed point of a self-referential equation involving the prime-counting function $\pi(x)$.

The logarithmic structure of both the RGE and the prime number theorem is not coincidental: the beta function is the derivative of the prime counting function in bilateral natural units.

2 The Prime Triple and Three Generations

The Bohr–Sommerfeld quantisation on S^3 gives three levels $y_n = n + 3/2$ for $n = 0, 1, 2$. The numerators of these levels are $\{3, 5, 7\}$.

Theorem 1 (Uniqueness of the Prime Triple)**.** *The set* $\{3, 5, 7\}$ *is the unique prime triple: the only three consecutive odd numbers that are all prime.*

Proof. Any three consecutive odd numbers have the form $\{n, n + 2, n + 4\}$. For $n \geq 5$, exactly one of $n, n+2, n+4$ is divisible by 3 (since among any three consecutive members of an arithmetic progression with common difference 2, the residues modulo 3 cycle through $\{0, 1, 2\}$). Therefore exactly one is composite. The only exception is $n = 3$, where 3 itself is the divisor rather than a multiple, giving $\{3, 5, 7\}$, all prime. □

This uniqueness forces the generation count to be exactly 3. The Bohr–Sommerfeld levels are $y_n = p_{n+1}/2$ for consecutive odd primes p_{n+1}, and there is only one such triple.

Remark 1. *The same number 3 appears as: the generation count, the Euler characteristic* $\chi(\mathbb{CP}^2) = 3$*, the Atiyah–Singer index of the Dirac operator on* $\mathbb{CP}^2$ *with SU(3) fundamental bundle, and the first element of the prime triple. All are the same topological fact.*

3 Yukawa Couplings as Prime Exponentials

3.1 The Formula

The bilateral crossing operation $\mathcal{B}$ maps the egress angular spectrum to the ingress face, producing the neutrino mass spectrum (derived in [1]). The same operation swaps the prime indices between the Koide eigenvalue labels and the Yukawa suppression exponents. With the Koide prefactors $K_{\text{eg}} = 2/3$ and $1/K_{\text{eg}} = 3/2$:

$$m_\tau = \frac{3}{2}\,\exp\!\left(-5 + \frac{4\alpha}{3}\right)\frac{v}{\sqrt{2}}, \tag{1}$$

$$m_\mu = \frac{2}{3}\,e^{-7}\,\frac{v}{\sqrt{2}}, \tag{2}$$

$$m_e = \text{Koide}(m_\tau,\, m_\mu). \tag{3}$$

The correction $+4\alpha/3$ in (1) is the leading-order QED anomalous dimension of the tau Yukawa coupling; 4/3 is the Casimir invariant of the SU(2) fundamental representation for a lepton.

Table 1: Lepton mass predictions vs. observation [6]

Lepton	Formula	Predicted (MeV)	Observed (MeV)
τ	$\frac{3}{2}\,e^{-(5-4\alpha/3)}\,v/\sqrt{2}$	1776.858	1776.860
μ	$\frac{2}{3}\,e^{-7}\,v/\sqrt{2}$	105.841	105.660
e	$\text{Koide}(m_\tau, m_\mu)$	0.5106	0.5110

3.2 Derivation from the RGE

The Yukawa coupling $Y(\mu)$ satisfies the one-loop RGE:

$$\frac{dY}{d\ln\mu} = -\gamma\,Y, \tag{4}$$

where γ is the anomalous dimension. The solution is:

$$Y(m_k) = Y(M_U)\,\exp\!\left(-\gamma\,\ln\frac{M_U}{m_k}\right) = Y_0\,e^{-\gamma\,L_k}, \tag{5}$$

where $L_k = \ln(M_U/m_k)$ is the integrated log-flow from the unification scale to generation k.

Theorem 2 (Bilateral Natural Units [2])**.** *The Yukawa anomalous dimension equals exactly 1 in bilateral natural units, and the integrated RGE flow to generation k equals the prime p_k labelling that generation:*

$$\gamma = 1, \qquad L_k = \ln(M_U/m_k) = p_k. \tag{6}$$

This follows from the bilateral action quantisation $S_{\text{bilateral}} = 4\pi$: one unit of RGE flow corresponds to one bilateral mode step, and the wavefunction overlap at mode n is suppressed by e^{-n}, giving $\gamma = 1$ [2]. The same 4π that gives $\alpha_U = 1/42$ via the instanton/Chern–Simons argument [1] also gives $\gamma = 1$ in the Yukawa sector.

The identification $L_k = p_k$ is equivalent to:

$$m_k = M_U \, e^{-p_k}, \tag{7}$$

i.e. each generation threshold is an exponential of the corresponding prime below the unification scale. The primes are $p_\tau = 5$ and $p_\mu = 7$ (after the bilateral swap of indices from the Koide eigenvalue labels).

3.3 The Prime Number Theorem as the RGE

The prime number theorem states that the average prime gap near p is $\ln p$. In bilateral natural units, consecutive generation thresholds are separated by:

$$\ln \frac{m_\mu}{m_\tau} \approx p_\tau - p_\mu = 5 - 7 = -2 \quad \Longrightarrow \quad \frac{m_\tau}{m_\mu} = e^{p_\mu - p_\tau} = e^2. \tag{8}$$

The observed ratio (with Koide prefactors) is:

$$\frac{m_\tau}{m_\mu} = \frac{(3/2)\, e^{-5}}{(2/3)\, e^{-7}} = \frac{9}{4}\, e^2 \approx 16.62. \tag{9}$$

The observed value is $m_\tau/m_\mu = 16.82$, a 1.2% agreement. The 9/4 prefactor is the ratio of the bilateral Koide prefactors; the e^2 is the exponential of the prime gap $7 - 5 = 2$.

More generally, the mass ratio between any two generations is $e^{\Delta p}$ times a ratio of Koide prefactors, where Δp is the prime gap between their exponents. By the prime number theorem, the average prime gap near p is $\ln p$, so the typical inter-generation mass ratio grows as $e^{\ln p} = p$ — a power-law hierarchy set by the prime index, not by a free parameter.

3.4 The Hierarchy Problem

The electron mass is $m_e \approx 0.511\,\text{MeV}$ while the Planck mass is $M_{\text{Pl}} \approx 1.22 \times 10^{19}\,\text{GeV}$. In the Standard Model, this ratio $m_e/M_{\text{Pl}} \approx 10^{-22}$ requires a fine-tuned Yukawa coupling $y_e \approx 3 \times 10^{-6}$.

In the bilateral framework:

$$y_e = \frac{m_e\sqrt{2}}{v} = \frac{2}{3}\, e^{-13} + (\text{Koide correction}), \tag{10}$$

where $p = 13$ is the prime at index 6 (the electron prime index after the bilateral crossing structure). The smallness of y_e is not fine-tuned: it is e^{-13}, the exponential of a prime. The desert between the electroweak and Planck scales is e^{-p} for a specific prime p. Primes are not fine-tuned; they are what the integers contain.

The electron mass is then fixed by the Koide relation given m_τ and m_μ: it requires no independent prime index. The Koide relation is the constraint that closes the three-mass system.

4 The U(1) Coupling as a Prime Fixed Point

4.1 Setup

The bilateral framework gives $\alpha_U = 1/42$, so $1/\alpha_U = 42 = N_{\text{gen}} \times \dim(S^3 \times \mathbb{CP}^2) = 3 \times 7$. The one-loop RGE for the U(1) coupling gives:

$$\frac{1}{\alpha_1(M_Z)} = \frac{1}{\alpha_U} + \frac{b_1}{2\pi} \ln \frac{M_U}{M_Z}, \tag{11}$$

with $b_1 = 41/10$ in the GUT normalisation. This running takes $1/\alpha_1$ from 42 at M_U to its observed value at M_Z.

4.2 The Prime Fixed Point

Theorem 3 (U(1) Fixed Point)**.** *The unique prime solution to the equation*

$$x - \frac{1}{\alpha_U} = \pi(x), \tag{12}$$

where $\pi(x)$ is the prime-counting function and $1/\alpha_U = 42$, is $x = 59$.

Proof. We seek integer solutions to $x - 42 = \pi(x)$ with x prime. Direct enumeration gives $x = 58$ ($\pi(58) = 16 = 58 - 42$, but $58 = 2 \times 29$ is composite) and $x = 59$ ($\pi(59) = 17 = 59 - 42$, and 59 is prime). No other solution exists in the range $[42, 200]$ by explicit computation, and for large x the PNT gives $\pi(x) \approx x/\ln x \ll x - 42$, so no further solutions exist. The unique prime solution is $x = 59$. □

Corollary 4. $1/\alpha_1(M_Z) = 59$*, i.e.* $\alpha_1(M_Z) = 1/59$*, in exact agreement with the observed value.*

The observed U(1) inverse coupling at M_Z is $1/\alpha_1 = 59.00$ (from $\sin^2\theta_W = 0.23122$ and $\alpha_{\text{em}}(M_Z) = 1/127.9$). The prediction $x = 59$ is exact.

4.3 Interpretation

Equation (12) states that the inverse coupling at M_Z minus the inverse unified coupling equals the number of primes below the inverse coupling:

$$\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_U} = \pi\left(\frac{1}{\alpha_1(M_Z)}\right). \tag{13}$$

This is self-referential: the *amount* of RGE running between M_U and M_Z equals the number of primes that fit below the final coupling value. The running is not a continuous flow parameterised by $\ln \mu$ but a discrete stepping, one step per prime. The prime-counting function is the integrated RGE flow, and the fixed point is the scale at which these two counts coincide.

The prime 59 is the inverse coupling because: starting from $1/\alpha_U = 42$, the coupling runs upward through 17 discrete prime steps, arriving at the 17th step above 42. The value at that step is $42 + \pi(59) = 42 + 17 = 59$. The fact that 59 is itself prime means the U(1) coupling lands precisely on a prime — not between primes — at the Z scale.

4.4 Connection to the Unification Scale

The U(1) RGE then gives the unification scale consistent with $b_1 = 41/10$:

$$\ln \frac{M_U}{M_Z} = \frac{(1/\alpha_1(M_Z) - 1/\alpha_U) \cdot 2\pi}{b_1} = \frac{17 \times 2\pi}{41/10} = \frac{340\pi}{41} \approx 26.05, \tag{14}$$

giving $M_U \approx 1.87 \times 10^{13}$ GeV. This matches the unification scale derived independently in the framework from the Riemann zero gap structure [1].

Note that $b_1 = 41/10$ where $41 = p_{13}$ (the 13th prime), and $340 = 20 \times 17 = 20 \times \pi(59)$. The logarithm of the hierarchy is $340\pi/41$: a ratio of primes times π. The appearance of π here reflects $g_1^2 = 2\pi \times (2/41)$; the $\alpha = g^2/(4\pi)$ convention removes it from the final coupling value.

5 Open Problems

1. SU(2) and SU(3) couplings. The U(1) fixed-point equation (12) extends to SU(2) and SU(3) via dimensional ratios of $S^3 \times \mathbb{CP}^2$: $1/\alpha_2 = 42 \times 5/7 = 30$ (exact) and $1/\alpha_s = 42/5 = 8.4$ (0.96% from observation). A unified prime equation governing all three couplings from a single formula is developed in [3].

2. The muon QED correction. The tau formula includes the leading-order QED anomalous dimension $4\alpha/3$, giving 0.0001% agreement. The corresponding correction for the muon is of opposite sign and smaller magnitude, consistent with the observed 0.17% discrepancy at tree level, but the explicit one-loop calculation for the muon has not been done.

3. Quark Yukawa hierarchies. The prime exponential structure for lepton masses has a natural extension to quarks with prime assignments $t(0), c(5), u(11)$ (up-type) and $b(3), s(7), d(11)$ (down-type). Quarks sit at non-prime positions in the Yukawa spectrum — within prime gaps — consistent with their confined nature [4]. The quark prefactors require the quark Koide relation and the colour sector of the bilateral framework, which remain open.

6 Conclusion

The lepton mass hierarchy is determined by the prime structure of the bilateral crossing geometry. The tau and muon Yukawa couplings are prime exponentials $Y_k = Y_0\, e^{-p_k}$ with Koide prefactors, where $p_k \in \{5, 7\}$ are primes from the unique prime triple $\{3, 5, 7\}$ after bilateral index swap. The tau mass is reproduced to 0.0001% once the leading QED anomalous dimension is included. The electron mass is fixed by the Koide closure relation and requires no independent prime index.

The Yukawa anomalous dimension $\gamma = 1$ is proved in [2] from the bilateral action quantisation $S_{\text{bilateral}} = 4\pi$: the same 4π that gives $\alpha_U = 1/42$ also gives $\gamma = 1$. Both are faces of the same bilateral identity.

The U(1) inverse coupling $1/\alpha_1(M_Z) = 59$ is the unique prime solution to $x - 42 = \pi(x)$, where $42 = 1/\alpha_U$ is the bilateral unified coupling. The SU(2) and SU(3) couplings are given by dimensional ratios of $S^3 \times \mathbb{CP}^2$ [3]. The running from M_U to M_Z is a discrete stepping through primes, with each step contributing one unit to the prime count. The prime number theorem and the renormalisation group equation have the same logarithmic structure because they are the same counting problem in different languages.

The hierarchy problem is dissolved: the mass scales are not fine-tuned but are set by specific primes. The desert between the electroweak and Planck scales is e^{-p} for a prime p. Primes are not numerology — they are the irreducible structure of the integer lattice, which the bilateral mesh identifies as the fundamental substrate of physical law.

References

[1] D. Low, *The Standard Model from a Bilateral Crossing Geometry*, preprint (2025), `ontologia.co.uk/standardmodel.html`.

[2] D. Low, *The Yukawa Anomalous Dimension in Bilateral Natural Units*, preprint (2025), `ontologia.co.uk/`.

[3] D. Low, *Gauge Coupling Constants from the Dimensional Structure of* $S^3 \times \mathbb{CP}^2$, preprint (2025), `ontologia.co.uk/`.

[4] D. Low, *Confinement as Incomplete Bilateral Crossing*, preprint (2025), `ontologia.co.uk/`.

[5] Y. Koide, *A fermion-boson composite model of quarks and leptons*, Phys. Lett. B **120** (1983) 161–165.

[6] Particle Data Group (S. Navas et al.), *Review of Particle Physics*, Phys. Rev. D **110** (2024) 030001.

[7] J. Hadamard, *Sur la distribution des zéros de la fonction* $\zeta(s)$, Bull. Soc. Math. France **24** (1896) 199–220.