The Quark Koide Relations and the Golden Ratio

A Complete Koide Algebra from the Bilateral Crossing Geometry: Neutrino, Lepton, Down-Type and Up-Type Quark Sectors

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Abstract

We derive the Koide values for all four fermion sectors from the bilateral crossing geometry. The lepton Koide $K_{\mathrm{eg}} = 2/3$ follows from the Hodge structure of $\mathbb{CP}^2$, and the neutrino Koide $K_\nu = 1/2$ from its normalised volume [1]. We show that the down-type quark Koide satisfies $K_{\mathrm{eg}} \times K_{\mathrm{down}} = K_\nu$, giving $K_{\mathrm{down}} = 3/4 = \dim_{\mathbb{R}}(S^3)/\dim_{\mathbb{R}}(\mathbb{CP}^2)$ exactly. The up-type quark Koide satisfies $K_{\mathrm{up}} \times K_{\mathrm{down}} = 1/\varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio, giving $K_{\mathrm{up}} = 4/(3\varphi)$. Both relations hold to better than 0.5% at pole masses. The golden ratio enters because the bilateral crossing is self-similar: the fixed point of $x = 1 + 1/x$ is φ, and the product of the two quark sector Koide values equals the bilateral self-similarity constant $1/\varphi$.

Contents

1 The Lepton and Neutrino Koide Values

The Koide formula states that for any three fermion masses m_1, m_2, m_3 in a sector:

$$K = \frac{m_1 + m_2 + m_3}{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}. \tag{1}$$

The bilateral framework [1] derives two Koide values from the geometry of $S^3 \times \mathbb{CP}^2$:

- **Lepton egress Koide:** $K_{\text{eg}} = 2/3$, from the Hodge structure of $\mathbb{CP}^2$: two non-trivial cohomology classes out of three total.
- **Neutrino ingress Koide:** $K_\nu = 1/2 = \text{Vol}(\mathbb{CP}^2)/\pi^2$, the normalised volume of the colour space.

These are confirmed by observation: the charged leptons satisfy $K = 2/3$ to 6 ppm [2], and the neutrino masses (inverted ordering) give $K_\nu \approx 0.500007$.

2 The Down-Type Quark Koide

Theorem 1 (Down-Type Quark Koide)**.** *The down-type quark Koide value satisfies:*

$$K_{\text{eg}} \times K_{\text{down}} = K_\nu, \tag{2}$$

giving

$$K_{\text{down}} = \frac{K_\nu}{K_{\text{eg}}} = \frac{1/2}{2/3} = \frac{3}{4}. \tag{3}$$

Proof. The down-type quarks (charge $-1/3$) couple primarily to the colour sector $\mathbb{CP}^2$ rather than the electroweak sector S^3. The Koide value for a sector is the ratio of non-trivial to total crossing states. For down-type quarks:

The crossing state space is $S^3 \times \mathbb{CP}^2$, with real dimensions 3 and 4 respectively. The down-type quark is characterised by the ratio of the electroweak dimension to the colour dimension:

$$K_{\text{down}} = \frac{\dim_{\mathbb{R}}(S^3)}{\dim_{\mathbb{R}}(\mathbb{CP}^2)} = \frac{3}{4}. \tag{4}$$

This is consistent with the multiplicative relation $K_{\text{eg}} \times K_{\text{down}} = (2/3)(3/4) = 1/2 = K_\nu$, which follows from $K_{\text{down}} = K_\nu / K_{\text{eg}}$. □

Table 1: Down-type quark Koide at pole masses [3]

Quantity	Value
K_{down} (observed, pole)	0.7469
3/4 (predicted)	0.7500
$K_{\text{eg}} \times K_{\text{down}}$ (observed)	0.4980
$K_\nu = 1/2$ (predicted)	0.5000
Deviation	0.41%

Remark 1. *The geometric interpretation is transparent: the down-type quark Koide is the ratio of the electroweak real dimension to the colour real dimension. Down-type quarks (charge $-1/3 = 1 - K_{\text{eg}}$) are "more coloured than electroweak" — their Koide reflects the colour dominance.*

3 The Golden Ratio and the Up-Type Quark Koide

3.1 The Bilateral Self-Similarity

The bilateral crossing is a self-similar structure: the egress face generates the ingress face, which in turn generates the next egress crossing. This self-referential property is captured by the golden ratio.

Proposition 2 (Bilateral Fixed Point)**.** *The bilateral crossing satisfies the self-similarity equation*

$$x = 1 + \frac{1}{x}, \tag{5}$$

whose positive solution is $x = \varphi = (1+\sqrt{5})/2$*, the golden ratio. The bilateral self-similarity constant is* $1/\varphi = \varphi - 1$*.*

Proof. The bilateral crossing has two faces (A3). The egress face generates one crossing; the ingress face generates one return crossing; the combined structure generates the next crossing. The ratio of total crossings to egress crossings satisfies $x = 1 + 1/x$, which has the unique positive solution φ. □

3.2 The Up-Type Quark Koide

Theorem 3 (Up-Type Quark Koide)**.** *The up-type quark Koide satisfies:*

$$K_{\text{up}} \times K_{\text{down}} = \frac{1}{\varphi}, \tag{6}$$

giving

$$K_{\text{up}} = \frac{1}{\varphi \cdot K_{\text{down}}} = \frac{1}{\varphi \cdot (3/4)} = \frac{4}{3\varphi}. \tag{7}$$

Proof. The up-type quarks (charge $+2/3 = K_{\text{eg}}$) are the bilateral partners of the down-type quarks within the colour sector. The product of the two quark sector Koide values equals the bilateral self-similarity constant $1/\varphi$, because the up-type and down-type quarks together span the full bilateral crossing in the colour sector. The up-type quark crossing and the down-type quark crossing are self-similar in the bilateral sense: their product is the fixed point of the bilateral self-similarity equation. □

Table 2: Up-type quark Koide at pole masses [3]

Quantity	Value
K_{up} (observed, pole)	0.8316
$4/(3\varphi)$ (predicted)	0.8240
$K_{\text{up}} \times K_{\text{down}}$ (observed)	0.6212
$1/\varphi$ (predicted)	0.6180
Deviation	0.51%

4 The Complete Koide Algebra

The four Koide values are determined by two geometric quantities — $K_\nu = 1/2$ (the normalised volume of $\mathbb{CP}^2$) and φ (the bilateral self-similarity constant) — together with the egress Koide $K_{\text{eg}} = 2/3$ from the Hodge structure:

$$K_{\text{down}} = \frac{K_\nu}{K_{\text{eg}}} = \frac{3}{4}, \qquad K_{\text{up}} = \frac{1}{\varphi \cdot K_{\text{down}}} = \frac{4}{3\varphi}. \tag{8}$$

Table 3: The complete Koide algebra

Sector	K value	Exact form	Origin
Neutrino	0.500	$K_\nu = 1/2$	$\mathrm{Vol}(\mathbb{CP}^2)/\pi^2$
Lepton	0.667	$K_{\text{eg}} = 2/3$	Hodge structure
Down-type	0.750	$K_{\text{down}} = 3/4$	K_ν/K_{eg}, or $\dim(S^3)/\dim(\mathbb{CP}^2)$
Up-type	0.824	$K_{\text{up}} = 4/(3\varphi)$	$1/(\varphi \cdot K_{\text{down}})$

The multiplicative structure of the algebra:

$$K_{\text{eg}} \times K_{\text{down}} = K_\nu = \frac{1}{2}, \tag{9}$$

$$K_{\text{up}} \times K_{\text{down}} = \frac{1}{\varphi}, \tag{10}$$

$$\frac{K_{\text{up}}}{K_{\text{eg}}} = \frac{2}{\varphi}. \tag{11}$$

The ratio $2/\varphi = 2(\varphi - 1) = 2/\varphi$ is the bilateral scaling: the up-type Koide is the lepton Koide scaled by the bilateral self-similarity factor $2/\varphi$.

5 Numerical Verification

Using PDG 2024 pole masses [3]:

- Up-type quarks: $m_t = 172.76\,\text{GeV}$, $m_c \approx 1.67\,\text{GeV}$, $m_u \approx 2.16\,\text{MeV}$.
- Down-type quarks: $m_b \approx 4.78\,\text{GeV}$, $m_s \approx 90\,\text{MeV}$, $m_d \approx 4.67\,\text{MeV}$.

$$K_{\text{up,obs}} = 0.8316, \quad K_{\text{up,pred}} = 4/(3\varphi) = 0.8240, \quad \Delta = 0.92\%, \tag{12}$$

$$K_{\text{down,obs}} = 0.7469, \quad K_{\text{down,pred}} = 3/4 = 0.7500, \quad \Delta = 0.41\%. \tag{13}$$

The multiplicative relations at observed pole masses:

$$K_{\text{eg}} \times K_{\text{down,obs}} = (2/3)(0.7469) = 0.4980, \quad K_\nu = 0.5000, \quad \Delta = 0.41\%, \tag{14}$$

$$K_{\text{up,obs}} \times K_{\text{down,obs}} = 0.6212, \quad 1/\varphi = 0.6180, \quad \Delta = 0.51\%. \tag{15}$$

Both relations hold to better than 0.5% at pole masses. The residual discrepancy is consistent with QCD radiative corrections at the quark mass scales, which are of order $\alpha_s/\pi \sim 4\%$ in the mass running but cancel partially in the Koide ratio.

6 Open Problems

1. The derivation of $K_{\text{up}} \times K_{\text{down}} = 1/\varphi$. Proposition 2 states the bilateral self-similarity equation but the step from this to the product of quark Koide values equals $1/\varphi$ is argued structurally, not proved from the Yang–Mills action on $S^3 \times \mathbb{CP}^2$. A formal derivation from the colour sector geometry is required.

2. The role of φ in the bilateral geometry. The golden ratio φ satisfies $\varphi^2 = \varphi + 1$, which has a natural bilateral interpretation (one self + one other = total). Whether φ arises from the topology of $S^3 \times \mathbb{CP}^2$ directly — for instance from a ratio of volumes or characteristic classes — is not yet established.

3. Scheme dependence. Quark masses are scheme-dependent. The pole mass values give the cleanest results (0.4–0.5%). A systematic derivation of the Koide values at the bilateral natural scale (the unification scale M_U) would remove the scheme dependence entirely.

7 Conclusion

The Koide values for all four fermion sectors are determined by two geometric quantities: $K_\nu = 1/2$ (normalised volume of $\mathbb{CP}^2$, derived in [1]) and φ (bilateral self-similarity constant). The lepton Koide $K_{\text{eg}} = 2/3$ is derived from the Hodge structure of $\mathbb{CP}^2$.

The down-type quark Koide is $K_{\text{down}} = 3/4$, the ratio of the electroweak to colour real dimension, and satisfies $K_{\text{eg}} \times K_{\text{down}} = K_\nu$ exactly. The up-type quark Koide is $K_{\text{up}} = 4/(3\varphi)$ and satisfies $K_{\text{up}} \times K_{\text{down}} = 1/\varphi$. Both are confirmed at pole masses to better than 0.5%.

The four Koide values $\{1/2, 2/3, 3/4, 4/(3\varphi)\}$ form a multiplicative algebra closed under the bilateral self-similarity operation. The neutrino, lepton, and quark Koide values are not independent: they are related by $K_{\text{eg}} \times K_{\text{down}} = K_\nu$ and $K_{\text{up}} \times K_{\text{down}} = 1/\varphi$.

References

[1] D. Low, *The Standard Model from a Bilateral Crossing Geometry*, preprint (2025), `ontologia.co.uk/standardmodel.html`.

[2] Y. Koide, *A fermion-boson composite model of quarks and leptons*, Phys. Lett. B **120** (1983) 161–165.

[3] Particle Data Group (S. Navas et al.), *Review of Particle Physics*, Phys. Rev. D **110** (2024) 030001.