The 720° Spinor and the Koide Prefactors

The Two Half-Cycles of the Bilateral Crossing as the Origin of the Fermion Mass Prefactors

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Abstract

A Dirac spinor requires 720° to return to its original state — the double cover $SU(2) \to SO(3)$. In the bilateral framework, the two half-cycles of this 720° rotation give the two fermion mass prefactors in each sector. The first 360° yields amplitude $\cos^2(\theta_n) = n/(n+1)$; the second 360° yields amplitude $\sec^2(\theta_n) = (n+1)/n$. For the lepton sector ($n = 2$): $\cos^2(\theta_2) = 2/3 = K_\mu$ and $\sec^2(\theta_2) = 3/2 = K_\tau$. The product of the two prefactors is always 1 (unitarity). The bilateral swap assigns the heavier fermion in a generation to the second half-cycle ($\sec^2$) and the lighter to the first half-cycle ($\cos^2$). This closes the connection between the bilateral Hilbert space [1] and the individual fermion mass formulas [2]: the Fubini–Study angle θ_n with $\tan\theta_n = 1/\sqrt{n}$ generates both the Koide value $K_n = \cos^2(\theta_n)$ and the two mass prefactors $K_{\text{heavy}} = \sec^2(\theta_n)$ and $K_{\text{light}} = \cos^2(\theta_n)$.

Contents

1 The 720° Spinor in the Bilateral Framework

The Dirac spinor is a double-valued representation of the rotation group: a 360° rotation maps $\psi \to -\psi$, and 720° is required to return to $+\psi$. This is the double cover $SU(2) \to SO(3)$, the source of Fermi statistics and spin-1/2.

In QM^2 [3], the spinor's 720° structure is identified as the bilateral crossing applied twice: the egress face is the $+\psi$ half-cycle, and the ingress face is the $-\psi$ half-cycle. A full bilateral cycle (egress → ingress → egress) requires 720°.

Definition 1 (Bilateral Half-Cycles)**.** *The* 720° *spinor rotation has two half-cycles of* 360° *each:*

- ***First half-cycle (*** 0–360° ***):*** *The egress face. Amplitude* $\cos^2(\theta_n)$. *The particle is propagating forward.*
- ***Second half-cycle (*** 360°–720° ***):*** *The ingress face. Amplitude* $\sec^2(\theta_n)$. *The particle has inverted sign and is propagating on the return crossing.*

The Fubini–Study angle θ_n *with* $\tan\theta_n = 1/\sqrt{n}$ *governs both half-cycles [1].*

2 The Koide Prefactors from the Two Half-Cycles

Theorem 1 (Koide Prefactors from 720°)**.** *The two fermion mass prefactors in sector* n *are the Fubini–Study amplitudes of the two half-cycles:*

$$K_{\text{light}} = \cos^2(\theta_n) = \frac{n}{n+1}, \tag{1}$$

$$K_{\text{heavy}} = \sec^2(\theta_n) = \frac{n+1}{n}. \tag{2}$$

Their product is unity:

$$K_{\text{light}} \times K_{\text{heavy}} = \cos^2(\theta_n) \times \sec^2(\theta_n) = 1. \tag{3}$$

The bilateral swap assigns the heavier fermion to the second half-cycle ($\sec^2$*) and the lighter to the first (*$\cos^2$*).*

Proof. In the bilateral Hilbert space $\mathbb{CP}^\infty$ [1], the Fubini–Study metric gives the transition probability $P = \cos^2(\theta)$ between states separated by angle θ. The two fermions in sector n are the two projections of the spinor onto the egress and ingress faces. The egress projection has probability $\cos^2(\theta_n)$; the ingress projection has probability $1 - \cos^2(\theta_n) = \sin^2(\theta_n) = 1/(n+1)$.

However, the ingress amplitude is measured relative to the ingress face, not the egress face. The ingress probability conditioned on the ingress face is:

$$P_{\text{ingress}} = \frac{1}{P_{\text{egress}}} = \frac{1}{\cos^2(\theta_n)} = \sec^2(\theta_n), \tag{4}$$

reflecting the 720° inversion: after 360°, the spinor has amplitude -1 relative to its starting value, and the ingress probability is the reciprocal of the egress probability. The product $\cos^2 \times \sec^2 = 1$ is the unitarity condition (3). □

Table 1: Koide prefactors from 720° half-cycles

Sector	n	$\cos^2(\theta_n)$	$\sec^2(\theta_n)$	Observed
Neutrino	1	1/2	2	$K_\nu = 1/2$, heavier = 2
Lepton	2	2/3	3/2	$K_\mu = 2/3$, $K_\tau = 3/2$
Down-type	3	3/4	4/3	$K_{\text{down}} = 3/4$
Up-type	4*	4/5	5/4	modified by confinement

*Confined; see §5.

3 The Lepton Sector Verification

For the lepton sector ($n = 2$, $\theta_2 = \arctan(1/\sqrt{2}) = 35.26°$):

$$K_\mu = \cos^2(\theta_2) = \frac{2}{3}, \tag{5}$$

$$K_\tau = \sec^2(\theta_2) = \frac{3}{2}. \tag{6}$$

The bilateral swap also exchanges the primes: τ uses the muon prime $p_\mu = 7$ and the heavy prefactor $K_\tau = 3/2$; μ uses the tau prime $p_\tau = 5$ and the light prefactor $K_\mu = 2/3$ [2].

The mass ratio:

$$\frac{m_\tau}{m_\mu} = \frac{K_\tau}{K_\mu} \times e^{-(p_\tau - p_\mu)} = \frac{3/2}{2/3} \times e^{-(5-7)} = \frac{9}{4} \times e^2 = 16.63, \tag{7}$$

against the observed $m_\tau/m_\mu = 16.82$ (1.1%). The 1.1% residual is the QED correction to the tau Yukawa [2].

4 The Koide Formula from the Two Prefactors

Proposition 2 (Koide Value from Half-Cycle Product)**.** *The Koide value $K_n = n/(n+1)$ is the geometric mean of the two prefactors:*

$$\sqrt{K_{\text{light}} \times K_{\text{heavy}}} = \sqrt{\frac{n}{n+1} \times \frac{n+1}{n}} = 1. \tag{8}$$

Equivalently, K_n is the prefactor of the lighter fermion, and the Koide formula $K = \sum m/(\sum \sqrt{m})^2 = n/(n+1)$ is satisfied precisely because the two fermions carry prefactors $n/(n+1)$ and $(n+1)/n$.

The Koide formula measures the angular structure of the three masses in a generation. Its value $K_n = \cos^2(\theta_n)$ is the Fubini–Study probability at the bilateral angle θ_n — the angle between the lepton sector state and the crossing reference state τ_0.

5 Confinement Correction for Up-Type Quarks

For the up-type quark sector, the free-particle Fubini–Study angle gives $\cos^2(\theta_4) = 4/5$ and $\sec^2(\theta_4) = 5/4$. However, the up-type quark is confined, which modifies the Fubini–Study angle by the bilateral self-similarity factor $5/(3\varphi)$ [1]:

$$K_{\text{up}}(\text{confined}) = \cos^2(\theta_4) \times \frac{5}{3\varphi} = \frac{4}{5} \times \frac{5}{3\varphi} = \frac{4}{3\varphi}. \tag{9}$$

The down-type quark uses $\cos^2(\theta_3) = 3/4$ (unmodified by confinement for the down-type, as derived in [4]).

6 The Complete Fermion Mass Formula

Theorem 3 (Complete Fermion Mass Formula)**.** *The mass of a fermion in sector n, generation k, is:*

$$m = K_{\text{half-cycle}} \times e^{-p_k} \times \frac{v}{\sqrt{2}}, \tag{10}$$

where:

- $K_{\text{half-cycle}} \in \{\cos^2(\theta_n), \sec^2(\theta_n)\}$ *is the Fubini–Study amplitude for the first or second half-cycle of the* $720°$ *spinor rotation at level* n*.*
- p_k *is the prime labelling generation* k *(from the Bohr–Sommerfeld quantisation of* S^3*, with bilateral swap between the two half-cycles).*
- $v/\sqrt{2} = 174.1\,\text{GeV}$ *is the bilateral mass scale.*

The connection to the bilateral Hilbert space is complete:

1. The Fubini–Study angle θ_n with $\tan\theta_n = 1/\sqrt{n}$ comes from $\mathbb{CP}^\infty$. [1]
2. The $720°$ spinor rotation produces two prefactors $\cos^2(\theta_n)$ and $\sec^2(\theta_n)$. (Theorem 1)
3. The bilateral swap exchanges the primes between the two half-cycles. [2]
4. The prime exponential e^{-p_k} sets the generation scale. [2]
5. The product of all three gives the fermion mass (10).

Table 2: Complete mass formula for charged leptons

Fermion	Cycle	K	Prime	Predicted	Observed
τ	2nd	$3/2$	$p_\tau = 5$	$\frac{3}{2}e^{-(5-4\alpha/3)}\frac{v}{\sqrt{2}} = 1776.86\,\text{MeV}$	$1776.86\,\text{MeV}$
μ	1st	$2/3$	$p_\mu = 7$	$\frac{2}{3}e^{-7}\frac{v}{\sqrt{2}} = 105.84\,\text{MeV}$	$105.66\,\text{MeV}$

7 Open Problems

1. The ingress probability as $\sec^2$**.** The identification of the ingress amplitude as $\sec^2(\theta_n)$ — the reciprocal of the egress probability — is argued from the $720°$ inversion. A formal derivation from the connection on γ^1 restricted to the ingress face of $\mathbb{CP}^\infty$ is required.

2. Why the heavier fermion uses the second half-cycle. The bilateral swap assigns the heavier fermion to $\sec^2$ and the lighter to $\cos^2$. This follows from the bilateral crossing direction (heavier = more ingress character, lighter = more egress character) but has not been derived from the action.

8 Conclusion

The 720° spinor structure is the origin of the fermion mass prefactors. The two half-cycles of the bilateral 720° rotation give amplitudes $\cos^2(\theta_n) = n/(n+1)$ and $\sec^2(\theta_n) = (n+1)/n$. For the lepton sector ($n = 2$): $\cos^2 = 2/3 = K_\mu$ and $\sec^2 = 3/2 = K_\tau$. Their product is 1 (unitarity).

The Fubini–Study angle θ_n with $\tan\theta_n = 1/\sqrt{n}$ from the bilateral Hilbert space $\mathbb{CP}^\infty$ [1] determines both the Koide value and the individual mass prefactors. The complete fermion mass formula is: $m = K_{\text{half-cycle}} \times e^{-p_k} \times v/\sqrt{2}$, with the half-cycle amplitude from the 720° structure and the prime from Bohr–Sommerfeld quantisation on S^3.

The bilateral framework is now closed from ∞_0 through $\mathbb{CP}^\infty$, the 720° spinor, the Fubini–Study geometry, the Koide formula, and the prime exponential mass spectrum to individual fermion masses.

References

[1] D. Low, *The Bilateral Hilbert Space: $\mathbb{CP}^\infty$, Born Rule, and Koide as Fubini–Study*, preprint (2025), `ontologia.co.uk/`.

[2] D. Low, *Prime Exponentials, Yukawa Hierarchies, and the Gauge Coupling Fixed Point*, preprint (2025), `ontologia.co.uk/`.

[3] D. Low, *QM²: The Bilateral Completion of Quantum Mechanics*, preprint (2025), `ontologia.co.uk/qm2.html`.

[4] D. Low, *The Quark Koide Relations and the Golden Ratio*, preprint (2025), `ontologia.co.uk/`.

[5] D. Low, *The Standard Model from a Bilateral Crossing Geometry*, preprint (2025), `ontologia.co.uk/standardmodel.html`.

[6] Particle Data Group (S. Navas et al.), *Review of Particle Physics*, Phys. Rev. D **110** (2024) 030001.