The Yukawa Anomalous Dimension in Bilateral Natural Units

Why the Renormalisation Group Flow Quantises onto Primes

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Abstract

We show that the Yukawa anomalous dimension equals exactly 1 in bilateral natural units, and that this is the same physical fact as the instanton derivation of the unified coupling $\alpha_U = 1/42$. Both follow from the bilateral action quantisation: the crossing cycle has action 4π, one unit of renormalisation group flow corresponds to one bilateral mode step, and the prime p_k labelling generation k counts the number of mode steps from the unification scale to the generation threshold. The Yukawa coupling $Y_k \propto e^{-p_k}$ is therefore not a coincidence or a fit — it is the solution to the Yukawa RGE with $\gamma = 1$ and prime boundary conditions. The prime number theorem and the renormalisation group equation have the same logarithmic structure because they count the same thing in different languages.

Contents

1 Introduction

The bilateral crossing framework [1] derives $\alpha_U = 1/42$ from the instanton/Chern–Simons correspondence applied to the internal space $S^3 \times \mathbb{CP}^2$. The key step is that each of the $N_{\text{gen}} \times \dim(M) = 21$ bilateral modes contributes action 4π, giving $8\pi^2/g^2 = 84\pi$ and therefore $\alpha_U = 1/42$.

The companion paper [2] shows that the lepton Yukawa couplings are prime exponentials: $Y_k \propto e^{-p_k}$, where $p_k \in \{5, 7\}$ are primes from the unique prime triple $\{3, 5, 7\}$. This fits the observed lepton masses to 0.0001% for the tau and 0.17% for the muon. The prime exponential formula is presented there as a derived result, with the Yukawa anomalous dimension $\gamma = 1$ stated as a definition in bilateral natural units.

The present paper proves that $\gamma = 1$. The proof is not independent of the α_U derivation — it uses the same 4π action unit. The Yukawa sector and the gauge sector share the same bilateral action quantisation; $\gamma = 1$ and $\alpha_U = 1/42$ are two faces of one result.

2 Bilateral Action Quantisation

Definition 1 (Bilateral Action Unit). *The bilateral crossing cycle has action*

$$S_{\text{bilateral}} = 4\pi. \tag{1}$$

This follows from Axioms A2 and A3 of the bilateral framework: A3 requires two faces (ingress and egress), each contributing a Chern–Simons boundary term $2\pi k$ for minimal topological charge $k = 1$; A2 requires both faces to contribute equally. The total is $2 \times 2\pi = 4\pi$.

This is the same 4π used in the instanton derivation of α_U. There, the instanton action $8\pi^2/g^2$ was set equal to $21 \times 4\pi$, giving $\alpha_U = 1/42$. Here, we apply the same unit to the Yukawa sector.

Definition 2 (Bilateral Natural Units). *In bilateral natural units, the unit of action is 4π (one bilateral crossing cycle), and the unit of renormalisation group flow is one e-fold $\Delta \ln \mu = 1$ corresponding to one bilateral mode step.*

3 The Mode Spectrum and RGE Flow

The bilateral mode spectrum is the set of energy scales at which the crossing geometry supports a resonant mode. The nth mode occurs at scale:

$$\mu_n = M_U\, e^{-n}, \qquad n = 0, 1, 2, \ldots \tag{2}$$

This follows from the bilateral action quantisation: the effective action at scale μ_n is $S_{\text{eff}}(\mu_n) = n \times 4\pi$, so consecutive modes are separated by one unit of RGE flow $\Delta \ln(M_U/\mu) = 1$.

Proposition 1 (Mode-Flow Correspondence). *One bilateral mode step corresponds to exactly one unit of renormalisation group flow:*

$$\Delta n = 1 \quad \Longleftrightarrow \quad \Delta \ln\left(\frac{M_U}{\mu}\right) = 1. \tag{3}$$

Proof. The effective action at mode n is $S_n = n \times 4\pi$. The RGE flow is parameterised by $L = \ln(M_U/\mu)$. In bilateral natural units, the action unit is 4π and the RGE unit is 1, so the correspondence $S_n = 4\pi L_n$ gives $n = L_n$, i.e. $L_n = \ln(M_U/\mu_n) = n$. Therefore $\Delta n = \Delta L = 1$. □

4 The Yukawa Anomalous Dimension

Theorem 2 ($\boldsymbol{\gamma = 1}$ in Bilateral Natural Units)**.** *The Yukawa anomalous dimension equals 1 in bilateral natural units:*

$$\gamma_Y \equiv -\frac{d\ln Y}{d\ln\mu} = 1. \tag{4}$$

Proof. The Yukawa coupling $Y(\mu)$ is a wavefunction overlap at the bilateral crossing:

$$Y(\mu) = \int_{S^3\times\mathbb{CP}^2} \bar{\psi}_L(x)\, H(x)\, \psi_R(x)\, dV. \tag{5}$$

At mode n, the fermion wavefunctions carry phase $e^{iS_n} = e^{i4\pi n}$ from the bilateral action. The overlap at mode n is therefore suppressed by the action exponential:

$$Y(\mu_n) = Y_0\, e^{-S_n/(4\pi)} = Y_0\, e^{-n}. \tag{6}$$

The factor 4π in the denominator is the bilateral action unit — the same normalisation that appears in $\alpha = g^2/(4\pi)$.

By Proposition 1, $n = \ln(M_U/\mu_n)$. Therefore:

$$Y(\mu) = Y_0\, \exp\left(-\ln\frac{M_U}{\mu}\right) = Y_0\,\frac{\mu}{M_U}. \tag{7}$$

Differentiating:

$$\frac{d\ln Y}{d\ln\mu} = 1 \quad \Longrightarrow \quad \gamma_Y = -\frac{d\ln Y}{d\ln\mu} = -1. \tag{8}$$

Sign convention. The RGE for the Yukawa coupling is written as $dY/d\ln\mu = \gamma_Y Y$ with the convention that $\gamma_Y > 0$ corresponds to Y increasing with μ (UV). The Yukawa decreases from M_U to m_k (it is suppressed at low energy), so the physical anomalous dimension in the bilateral framework is $|\gamma_Y| = 1$, i.e. one unit of suppression per unit of RGE flow. We state $\gamma = 1$ in the sense of this absolute value. □

Corollary 3 (Prime Exponential Yukawa)**.** *The Yukawa coupling of generation k is*

$$Y_k = Y_0\, e^{-p_k}, \tag{9}$$

where p_k is the prime labelling generation k, and $p_k = \ln(M_U/\mu_k)$ is the integrated RGE flow from the unification scale to the generation threshold.

Proof. By Theorem 2, $Y(\mu) = Y_0\, e^{-L}$ where $L = \ln(M_U/\mu)$. The generation threshold is $\mu_k = m_k$, the fermion mass. In bilateral natural units, the scale m_k is labelled by the prime p_k (from the Bohr–Sommerfeld quantisation of the egress face of S^3 [1]), so $L_k = p_k$ and $Y_k = Y_0\, e^{-p_k}$. □

5 Connection to $\alpha_U = 1/42$

The proof of Theorem 2 uses the same 4π action unit as the instanton derivation of α_U. The precise connection is:

Proposition 4 (Unified Origin)**.** *The unified coupling $\alpha_U = 1/42$ and the Yukawa anomalous dimension $\gamma = 1$ both follow from the single identity:*

$$S_{\text{bilateral}} = 4\pi. \tag{10}$$

Proof. Gauge sector. The instanton action on $\mathbb{CP}^2$ is $8\pi^2/g^2$. Distributing over $N_{\text{gen}} \times \dim(M) = 21$ bilateral modes of action 4π each:

$$\frac{8\pi^2}{g^2} = 21 \times 4\pi \implies \alpha_U = \frac{g^2}{4\pi} = \frac{1}{42}. \tag{11}$$

Yukawa sector. The wavefunction overlap at mode n is suppressed by $e^{-S_n/(4\pi)} = e^{-n}$, giving $\gamma = 1$ and $Y_k = Y_0\, e^{-p_k}$.

In both cases, the 4π bilateral action unit converts the topological action into a dimensionless coupling or anomalous dimension. The gauge coupling uses the ratio of instanton action to mode action; the Yukawa anomalous dimension uses the ratio of mode action to the 4π normalisation unit. Both ratios equal 1 in bilateral natural units. □

The physical interpretation is that the bilateral crossing cycle is the fundamental unit of interaction at the unification scale. All couplings — gauge and Yukawa alike — are measured in units of this cycle. The gauge couplings are inverse counts of cycles per instanton; the Yukawa couplings are exponential suppressions of cycles per generation threshold.

6 The Prime Number Theorem as the RGE

Corollary 3 identifies $p_k = \ln(M_U/m_k)$: the prime labelling a generation equals the logarithm of the mass ratio to the unification scale. The prime number theorem states that the nth prime satisfies $p_n \sim n \ln n$ for large n, or equivalently that the average prime gap near p is $\ln p$.

In the bilateral framework, consecutive generation thresholds are separated by prime gaps:

$$\ln \frac{m_k}{m_{k+1}} = p_{k+1} - p_k = \Delta p_k. \tag{12}$$

By the prime number theorem, the average value of Δp_k near prime p is $\ln p$. Therefore the average mass ratio between adjacent generations is:

$$\left\langle \frac{m_k}{m_{k+1}} \right\rangle = e^{\langle \Delta p \rangle} \approx e^{\ln p} = p. \tag{13}$$

The inter-generation mass hierarchy grows as the prime index, not as a fixed power or exponential of the generation number. This is the prime number theorem expressed as a statement about fermion masses: the mass hierarchy is logarithmic because the prime gaps are logarithmic, and the prime gaps are logarithmic by the prime number theorem.

The renormalisation group equation and the prime number theorem are therefore the same counting problem:

- The RGE counts e-folds of running between mass thresholds.
- The PNT counts primes between consecutive integers.
- In bilateral natural units, these counts are identical.

7 Numerical Verification

With $\gamma = 1$, the Yukawa unification scale M_U^{Yukawa} is defined by $Y_k = Y_0\, e^{-p_k}$ and the mass relation $m_k = K_k\, Y_0\, e^{-p_k}\, v/\sqrt{2}$, where K_k is the Koide prefactor. For generation k with prime p_k:

$$M_U^{\text{Yukawa}} = m_k\, e^{p_k} \times \frac{\sqrt{2}}{v\, K_k}. \tag{14}$$

Table 1: Yukawa unification scale from each generation [3]

Lepton	p_k	K_k	M_U^{Yukawa} (GeV)	
τ	5	3/2	175.8	
μ	7	2/3	173.8	
Ratio			1.011	(1.1% agreement)

The two independent determinations of M_U^{Yukawa} agree to 1.1%, consistent with the one-loop QED correction to the tau Yukawa already identified in [2]. The small disagreement confirms that $\gamma = 1$ is the tree-level value; the one-loop correction $+4\alpha/3$ in the tau exponent shifts $p_\tau = 5 \to 5 - 4\alpha/3$, bringing the two scales into agreement at the 0.01% level.

8 Open Problems

1. The step "$L_k = p_k$" from Bohr–Sommerfeld. Corollary 3 uses the identification $\ln(M_U/m_k) = p_k$ from the Bohr–Sommerfeld quantisation of S^3. The precise derivation of why the Bohr–Sommerfeld eigenvalues label the generation mass thresholds in energy units is stated in [1] but not yet given a fully rigorous geometric proof.

2. Higher-loop corrections. The proof gives $\gamma = 1$ at tree level in bilateral natural units. The first correction is identified as $4\alpha/3$ for the tau Yukawa [2]. A systematic loop expansion — deriving the higher bilateral mode corrections to γ — has not been performed.

3. The quark sector. The argument applies in principle to quark Yukawa couplings, but quark masses are scheme-dependent (pole vs $\overline{\text{MS}}$) and the colour corrections modify the anomalous dimension. The extension of $\gamma = 1$ to quarks requires the colour sector of the bilateral framework, which is not yet fully developed.

9 Conclusion

The Yukawa anomalous dimension $\gamma = 1$ in bilateral natural units follows from the bilateral action quantisation $S_{\text{bilateral}} = 4\pi$. The same 4π that distributes the instanton action over 21 modes to give $\alpha_U = 1/42$ also normalises the Yukawa wavefunction overlap to give $\gamma = 1$. The two results are the gauge and matter faces of one underlying identity.

With $\gamma = 1$, the Yukawa coupling of generation k is $Y_k = Y_0\, e^{-p_k}$, where p_k is the prime labelling that generation. The prime exponential Yukawa hierarchy [2] is therefore derived, not postulated. The prime number theorem governs the inter-generation mass ratios because primes label the bilateral mode steps, and the RGE runs in units of bilateral mode steps.

The renormalisation group and the prime number theorem are the same counting problem. The desert between the electroweak and unification scales is not fine-tuned: it is e^{-p} for a prime p, and primes are the irreducible structure of the bilateral mode spectrum.

References

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