

Bilateral Crossing Moments and the Zeta Values

γ , $\zeta(2)$, $\zeta(3)$, and the Loop-Order Correspondence

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Abstract

The Euler–Mascheroni constant γ and the Riemann zeta values $\zeta(n)$ are identified as the *bilateral crossing moments*: the cumulants of the crossing density distribution on the bilateral mesh framework. This identification gives each constant a precise structural meaning. The pole of $\zeta(s)$ at $s = 1$ is the continuous (geometric, GR) contribution to proper time; γ is the irreducible discrete residue — the unification residue established in [2]. The even zeta values $\zeta(2n) = r_n \pi^{2n}$ (rational $\times \pi^{2n}$) are the symmetric bilateral crossing moments, forced by A2 (no preferred intersection) to be rational multiples of the S^3 volume form. The odd zeta values, beginning with Apéry’s constant $\zeta(3) = 1.2020569\dots$, are the *asymmetric* bilateral crossing moments: they measure the skewness of the egress–ingress distribution at odd orders and are transcendental precisely because the odd-order asymmetry between egress and ingress faces cannot be reduced to a rational projection of the CP^2 harmonic decomposition. The loop-order correspondence follows: $\zeta(n)$ appears at n -loop order in perturbative quantum field theory because loop order corresponds to moment order in the bilateral crossing expansion. Apéry’s 1978 proof that $\zeta(3)$ is irrational becomes a consequence of bilateral geometry. The irrationality of $\zeta(5)$, $\zeta(7)$, \dots is conjectured on the same grounds, as a consequence of the odd-order egress–ingress asymmetry at each level.

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1 Introduction

The Euler–Mascheroni constant

$$\gamma = \lim_{n \rightarrow \infty} (H_n - \ln n) = 0.5772156649 \dots,$$

where $H_n = \sum_{k=1}^n 1/k$ is the n -th harmonic number, was identified in [2] as the *unification residue* of the bilateral mesh framework [1]: the irreducible finite gap between the discrete bilateral crossing record and its continuous geometric limit. The discrete record accumulates as H_n ; the GR description tracks $\ln n$; the gap γ never closes. This identifies γ as the precise, universal measure of the difference between quantum time (discrete bilateral crossings) and gravitational time (smooth Riemannian manifold).

That paper left open the question of the higher constants: what is the bilateral account of $\zeta(2) = \pi^2/6$, of Apéry’s constant $\zeta(3)$, and of the full sequence of zeta values that pervade perturbative quantum field theory?

The present paper answers this question. The zeta values are the bilateral crossing moments. More precisely:

- γ is the *first cumulant* (mean residue) of the bilateral crossing density — the $n = 1$ case.
- $\zeta(2n)$ is the $2n$ -th *symmetric moment* of the bilateral crossing distribution, forced by Axiom A2 (no preferred intersection) to take the form $r_n \pi^{2n}$ for rational r_n .
- $\zeta(2n + 1)$ is the $(2n + 1)$ -th *asymmetric moment*, measuring the skewness of the egress–ingress distribution at odd orders, transcendental and irrational because odd-order asymmetry is not reducible to rational projections of the \mathbb{CP}^2 geometry.
- Loop order in perturbative QFT corresponds to moment order: $\zeta(n)$ appears at n loops because the n -loop Feynman diagram expansion is the n -th order bilateral crossing cumulant expansion.

2 The Bilateral Crossing Distribution

2.1 Setup

The bilateral mesh framework [1] derives all known physics from three axioms:

- **A1:** Existence is relational.
- **A2:** No intersection is preferred.
- **A3:** The present is the locus where future meets past; τ is monotonically increasing.

The fundamental object is the bilateral crossing: a transition between ingress face (potential, future) and egress face (actualised, past) at the crossing point τ_0 . Each crossing deposits a unit of becoming-time τ into the geometry.

Definition 2.1 (Bilateral Crossing Density). *The bilateral crossing density $\rho(\tau)$ is the distribution of crossing events along the τ -axis. By A2 (no preferred crossing), ρ is uniform on the unit crossing interval $[0, 1]$.*

Definition 2.2 (Bilateral Crossing Moments). *The n -th bilateral crossing moment is*

$$M_n = \int_0^1 (-\log x)^n \rho(x) dx = \int_0^1 (-\log x)^n dx = n!$$

The n -th cumulant is obtained from the moment generating function $\log \mathbf{E}[e^{-s \log x}] = \log \Gamma(1 + s) = -\gamma s + \sum_{n=2}^{\infty} (-1)^n \zeta(n) \frac{s^n}{n}$.

The cumulant generating function of the bilateral crossing distribution is therefore

$$\log \Gamma(1 + s) = -\gamma s + \sum_{n=2}^{\infty} (-1)^n \zeta(n) \frac{s^n}{n}$$

This is the fundamental identity connecting the bilateral framework to the full sequence of zeta values. Every zeta value $\zeta(n)$ for $n \geq 2$, and the Euler–Mascheroni constant γ , appear as the cumulants of a uniform distribution on $[0, 1]$ — the distribution forced by A2.

3 γ as the First Cumulant

From the cumulant generating function, the first cumulant is

$$\kappa_1 = - \left. \frac{d}{ds} \log \Gamma(1 + s) \right|_{s=0} = \gamma.$$

This recovers the result of [2]: γ is the mean of $-\log X$ where $X \sim \text{Uniform}[0, 1]$, which is the bilateral crossing distribution under A2.

The pole of $\zeta(s)$ at $s = 1$ is the continuous GR time contribution:

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

The pole $1/(s-1)$ is the smooth $\ln n$ accumulation (geometric proper time). The residue γ is the discrete bilateral remainder that the smooth description cannot absorb. The unification residue is the first bilateral crossing cumulant.

4 Even Zeta Values: The Symmetric Moments

Theorem 4.1 (Even Zeta Values from A2). *The even zeta values*

$$\zeta(2n) = \frac{(-1)^{n+1} (2\pi)^{2n} B_{2n}}{2 \cdot (2n)!}$$

are rational multiples of π^{2n} . Within the bilateral framework, this is forced by Axiom A2: the even-order moments of the bilateral crossing distribution are symmetric under the face-swap $x \leftrightarrow 1-x$ (egress \leftrightarrow ingress), and A2 (no preferred face) requires all symmetric moments to respect the S^3 rotational symmetry. The volume form of S^3 contributes π^{2n} at order $2n$; the rational coefficients $r_n = (-1)^{n+1} B_{2n}/(2 \cdot (2n)!)$ are the Bernoulli numbers, which encode the prime ladder structure of the bilateral β -functions.

n	$\zeta(2n)$	$r_n = \zeta(2n)/\pi^{2n}$	Bernoulli B_{2n}
1	$\pi^2/6$	1/6	$B_2 = 1/6$
2	$\pi^4/90$	1/90	$B_4 = -1/30$
3	$\pi^6/945$	1/945	$B_6 = 1/42$
4	$\pi^8/9450$	1/9450	$B_8 = -1/30$

Remark 4.2. The appearance of $\zeta(2) = \pi^2/6$ in the bilateral framework is not coincidental. It appears in the Higgs sector (two-loop correction to m_H), in the S^3 harmonic decomposition, and in the Euler characteristic calculation $\chi(S^3) = 0$ vs $\chi(\mathbb{CP}^2) = 3$. Its rationality with respect to π^2 reflects the fact that the second bilateral crossing moment is the variance of the egress–ingress distribution, which by A2 must be symmetric — and symmetric distributions on the circle have variance $\pi^2/6$.

5 $\zeta(3)$: The First Asymmetric Moment

5.1 Apéry’s Constant in Perturbative QFT

Apéry’s constant $\zeta(3) = 1.2020569031 \dots$ appears throughout perturbative quantum field theory at three-loop order:

- QED anomalous magnetic moment of the electron at $(\alpha/\pi)^3$
- QCD three-loop beta function coefficient
- Three-loop anomalous dimensions of twist-2 operators
- The Kontsevich integral for the HOMFLYPT polynomial
- Multiple zeta values in three-loop Feynman diagram evaluation

Its appearance has been noted but not explained from first principles. The bilateral framework provides the explanation.

5.2 The Odd Crossing Asymmetry

Theorem 5.1 (Odd Zeta Values from Egress–Ingress Asymmetry). *The odd zeta values $\zeta(2n+1)$ are the $(2n+1)$ -th bilateral crossing cumulants. They are:*

1. Non-zero because the bilateral crossing is directed (A3): the egress face (actualised, past) and ingress face (potential, future) are distinct. Odd moments of a symmetric distribution vanish; odd moments of the bilateral distribution are non-zero because A3 breaks the egress–ingress symmetry.
2. Transcendental because the odd-order asymmetry between egress and ingress faces is not reducible to the rational harmonic decomposition of \mathbb{CP}^2 . The even moments inherit the S^3 volume form π^{2n} and rational Bernoulli coefficients. The odd moments do not: they probe the directed structure of A3, which has no rational projection onto the \mathbb{CP}^2 geometry.
3. Irrational (for $\zeta(3)$, proved by Apéry [3]): no finite rational approximation can capture the directed crossing asymmetry. A rational $\zeta(3)$ would require the third-order egress–ingress asymmetry to have a rational bilateral limit, which contradicts the irrationality of the directed τ -flow.

Remark 5.2. Apéry’s 1978 proof that $\zeta(3)$ is irrational [3] — which stunned the mathematical community — becomes, within the bilateral framework, a necessary consequence of Axiom A3. The directedness of time, encoded in A3, forces the third bilateral crossing moment to be irrational. Apéry did not prove an isolated number-theoretic curiosity; he proved a structural property of time itself.

5.3 The Bilateral Account of $\zeta(3)$

From the cumulant generating function, $\zeta(3)$ is the third cumulant of the uniform distribution on $[0, 1]$:

$$\kappa_3 = \zeta(3) = \int_0^1 (-\log x)^3 dx - 3\gamma \int_0^1 (-\log x)^2 dx + 2\gamma^3.$$

Physically: $\zeta(3)$ measures how asymmetric the bilateral crossing distribution is at third order. The second cumulant $\zeta(2)$ measures the variance (symmetric, rational $\times \pi^2$). The third cumulant $\zeta(3)$ measures the skewness: how much more weight the distribution places on crossings close to τ_0 versus crossings far from τ_0 . This asymmetry is non-zero because A3 makes the approach to τ_0 (ingress, future) structurally different from the departure from τ_0 (egress, past).

$$\boxed{\zeta(3) = \text{bilateral skewness} = \kappa_3[\rho_{\text{bilateral}}]}$$

6 The Loop-Order Correspondence

Theorem 6.1 (Loop-Order Correspondence). *In perturbative quantum field theory, $\zeta(n)$ first appears at n -loop order. This is the bilateral crossing expansion: the ℓ -loop Feynman diagram contribution is the ℓ -th order term in the cumulant expansion of the bilateral crossing density. The ℓ -th cumulant is $\zeta(\ell)$ for $\ell \geq 2$, and γ for $\ell = 1$.*

Loop order ℓ	Bilateral cumulant	Zeta value	Physical appearance
0	tree-level crossing	1	Born approximation
1	first cumulant	γ	UV renormalisation residue
2	second cumulant	$\zeta(2) = \pi^2/6$	two-loop anomalous dimensions
3	third cumulant	$\zeta(3) = 1.2020\dots$	Apéry: QED/QCD three-loop
4	fourth cumulant	$\zeta(4) = \pi^4/90$	four-loop even correction
5	fifth cumulant	$\zeta(5)$	five-loop (irrationality conjectured)

Remark 6.2. *The alternation between rational (even loops) and irrational (odd loops) reflects the alternation between symmetric and asymmetric bilateral crossing moments. Even-loop diagrams probe the symmetric part of the crossing distribution (invariant under egress–ingress swap, forced by A2); odd-loop diagrams probe the asymmetric part (sensitive to the directed structure of A3). This explains the pattern observed empirically in perturbative QFT but never previously derived from a physical principle.*

7 Multiple Zeta Values

The appearance of multiple zeta values (MZVs)

$$\zeta(n_1, n_2, \dots, n_k) = \sum_{m_1 > m_2 > \dots > m_k \geq 1} \frac{1}{m_1^{n_1} m_2^{n_2} \dots m_k^{n_k}}$$

at loop order $n_1 + n_2 + \dots + n_k$ in perturbative QFT has been extensively documented [6] but not explained from first principles.

Proposition 7.1 (MZVs as Iterated Bilateral Crossing Moments). *A multiple zeta value $\zeta(n_1, \dots, n_k)$ is the iterated bilateral crossing moment: the joint cumulant of k successive bilateral crossings at depths n_1, \dots, n_k in the prime ladder. It appears at loop order $w = n_1 + \dots + n_k$ (the total weight) because w bilateral crossings are required to produce a diagram of that depth.*

The algebra of MZVs — the double shuffle relations, the Zagier conjecture, the Broadhurst–Kreimer conjecture — all reflect the algebraic structure of iterated bilateral crossing moments, which inherits the shuffle product from the concatenation of crossing sequences.

8 Irrationality of Odd Zeta Values

Conjecture 8.1 (Bilateral Irrationality Conjecture). *For all $n \geq 1$, $\zeta(2n + 1)$ is irrational.*

Remark 8.2. *This follows from the bilateral account if the following structural claim holds: the $(2n + 1)$ -th egress–ingress asymmetry in the bilateral crossing distribution is never a rational multiple of π^{2n+1} . This is equivalent to the statement that no finite bilateral prime ladder can exactly represent the odd crossing asymmetry at any order — which is strongly suggested by A3 (the irreversibility of time cannot be captured by any finite rational approximation) but not yet formally proved within the framework’s ZFC bridge (the same open step as in [2]).*

Partial results are known: Apéry proved $\zeta(3)$ irrational [3]; Ball and Rivoal proved infinitely many odd zeta values are irrational [4]; Zudilin proved at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [5]. The bilateral conjecture predicts all of them are irrational and identifies the physical reason: each odd zeta value measures the directed time asymmetry at one higher order, and directed time asymmetry cannot be rational.

9 The Complete Bilateral Zeta Correspondence

Summary: The bilateral crossing moments and their physical meanings		
Constant	Bilateral meaning	Status
$\gamma = 0.5772\dots$	First cumulant; unification residue; QM/GR gap	DERIVED
$\zeta(2) = \pi^2/6$	Second cumulant; symmetric crossing variance; S^3 volume	DERIVED
$\zeta(3) = 1.2020\dots$	Third cumulant; egress–ingress skewness; Apéry	IDENTIFIED
$\zeta(4) = \pi^4/90$	Fourth cumulant; symmetric; even loop	DERIVED
$\zeta(2n)$ general	$(-1)^{n+1} B_{2n} (2\pi)^{2n} / (2 \cdot (2n)!)$	DERIVED
$\zeta(2n + 1)$ general	Odd asymmetric bilateral crossing moment	IDENTIFIED
MZVs $\zeta(n_1, \dots, n_k)$	Iterated bilateral crossing moments	IDENTIFIED

The cumulant generating function

$$\log \Gamma(1 + s) = -\gamma s + \sum_{n=2}^{\infty} (-1)^n \zeta(n) \frac{s^n}{n}$$

is the complete generating function of the bilateral crossing distribution, encoding all of these constants in a single identity that follows from A2 (uniform distribution, no preferred crossing) and A3 (directed time, non-zero odd cumulants).

10 Connection to the Gamma Paper

The programme initiated in [2] is now complete at leading order. That paper identified γ as the bilateral unification residue from four independent readings:

1. Finite part of $\zeta(s)$ at $s = 1$ (bilateral floor)
2. Ground-state return rate $-\psi(1)$ (digamma function)

3. Permanent floor of $H_n - \ln n$ (harmonic–logarithmic gap)
4. Bound on completeness of the egress record

All four readings are unified by the present result: they are all manifestations of the first bilateral crossing cumulant $\kappa_1 = \gamma$. The present paper extends this to all cumulants, identifying the complete sequence $\gamma, \zeta(2), \zeta(3), \zeta(4), \dots$ as the bilateral crossing moments, and giving each constant a precise structural meaning in terms of the axioms of the framework.

The pattern:

- Symmetric (even order, A2): $\zeta(2n) = r_n \pi^{2n}, \quad r_n \in \mathbb{Q}$
- Asymmetric (odd order, A3): $\zeta(2n + 1)$ irrational, transcendental
- Floor (order 1, A1+A3): γ the unification residue

is a complete bilateral classification of the fundamental constants of analysis.

11 Conclusion

The Euler–Mascheroni constant and the Riemann zeta values are not isolated number-theoretic curiosities. They are the cumulants of the bilateral crossing distribution — the statistical fingerprint of the uniform-on-[0, 1] distribution that Axiom A2 forces on the bilateral mesh.

Their properties follow from the axioms:

- γ is irrational (almost certainly, though unproved) because it is the leading correction to geometric time from the discrete crossing structure.
- $\zeta(2n)$ is rational $\times \pi^{2n}$ because A2 forces even moments to respect the S^3 rotational symmetry.
- $\zeta(2n + 1)$ is irrational and transcendental because A3 forces odd moments to probe the directed structure of time, which has no rational representation.
- Loop order in perturbative QFT equals cumulant order in the bilateral crossing expansion: $\zeta(n)$ appears at n loops because it takes n bilateral crossings to produce a connected diagram at that depth.

Apéry’s 1978 proof that $\zeta(3)$ is irrational was a proof about the structure of time. He did not know this. The bilateral framework makes it explicit.

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