The Uniqueness of $S^3 \times \mathbb{CP}^2$

Four Bilateral Constraints Force the Geometry

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Abstract

We prove that $S^3 \times \mathbb{CP}^2$ is the unique compact Riemannian 7-manifold satisfying four bilateral constraints: (A) a non-trivial spinor double cover compatible with the 720° bilateral cycle [3]; (B) a Fubini–Study metric giving the Koide sequence $K_n = n/(n+1)$ [2]; (C) isometry group exactly $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$; and (D) minimal dimension 7. Each constraint is necessary and each eliminates a class of competitors. The proof uses Perelman's geometrisation theorem for constraint A, the classification of compact Kähler manifolds with SU(3) isometry for constraint B, and the classification of compact 7-manifolds for constraint D. Additionally, the three fermion generations are identified with the three non-vanishing even cohomology classes $H^0(\mathbb{CP}^2)$, $H^2(\mathbb{CP}^2)$, $H^4(\mathbb{CP}^2)$, giving $\chi(\mathbb{CP}^2) = 3$ as the topological origin of the generation number.

Contents

1 Introduction

The bilateral crossing framework [1] identifies the internal geometry of the universe as $S^3 \times \mathbb{CP}^2$, producing the Standard Model gauge group and particle content. A natural question is whether this identification is unique: could any other compact manifold serve the same role?

This paper answers the question. Four constraints, each arising directly from the bilateral axioms, combine to force $S^3 \times \mathbb{CP}^2$ uniquely. No other compact manifold satisfies all four simultaneously.

2 The Four Constraints

Definition 1 (Bilateral Manifold)**.** *A* bilateral manifold *is a compact Riemannian manifold M satisfying:*

(A) ***Spinor double cover:*** *The spin structure on M is non-trivial and the spinor bundle returns to itself only after a 720 rotation (the bilateral double cover).*

(B) ***Koide–Fubini–Study:*** *The natural metric on M is the Fubini–Study metric giving $K_n = n/(n+1)$ for $n = 1, 2, 3, \ldots$*

(C) ***Standard Model isometry:*** *The isometry group of M is exactly* $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

(D) ***Minimal dimension:*** $\dim(M)$ *is the smallest integer for which constraints A–C can be simultaneously satisfied.*

3 Constraint A: The 720° Spinor Forces S^3

Theorem 1 (Spinor Constraint)**.** *The component of M carrying the bilateral spinor structure is diffeomorphic to S^3.*

Proof. The 720° bilateral cycle requires a base manifold B with the following properties:

- B is compact (the crossing is periodic).
- B is simply connected: $\pi_1(B) = 0$ (the Möbius twist is the *only* non-contractible loop, arising from the spinor double cover, not from π_1).
- B admits a free action of $\mathrm{Spin}(3) = \mathrm{SU}(2)$ (the spinor structure group).
- $\dim(B) = 3$ (the minimal dimension for a non-trivial spin structure with $\mathrm{SU}(2)$ symmetry).

By Perelman's geometrisation theorem [5], every simply-connected compact 3-manifold is diffeomorphic to S^3. Therefore $B \cong S^3$. □

Remark 1. *The condition $\pi_1(B) = 0$ is physically necessary: if B had non-trivial fundamental group, there would be additional non-contractible loops beyond the spinor double cover, giving additional topological superselection sectors not present in the Standard Model.*

4 Constraint B: The Koide Sequence Forces $\mathbb{CP}^2$

Theorem 2 (Koide–Fubini–Study Constraint). *The component of* M *carrying the Fubini–Study structure giving* $K_n = n/(n+1)$ *is* $\mathbb{CP}^2$.

Proof. The Koide sequence $K_n = \cos^2(\theta_n) = n/(n+1)$ with $\tan(\theta_n) = 1/\sqrt{n}$ arises from the Fubini–Study metric on $\mathbb{CP}^\infty$ [2]. The finite-dimensional truncation carrying K_1, K_2, K_3 must be a compact Kähler manifold C with:

- Fubini–Study metric (so the Koide values are well-defined as geodesic distances).
- Isometry group containing SU(3) (for the colour force, as argued in §5).
- $\mathbb{C}$-dimension $k \geq 2$ (to accommodate at least two Koide levels for the two quark chiralities).

The compact Hermitian symmetric spaces with SU(3) isometry are classified:

- $\mathbb{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2)$: isometry group $\mathrm{PU}(3) \supset \mathrm{SU}(3)$, real dimension 4.
- $\mathbb{CP}^k$ for $k > 2$: isometry group $\mathrm{PU}(k+1) \supset \mathrm{SU}(k+1) \supset \mathrm{SU}(3)$, but with additional symmetry beyond SU(3) giving spurious gauge bosons not present in the Standard Model.

Only $\mathbb{CP}^2$ has isometry group with exactly SU(3) as its simple factor. Therefore $C = \mathbb{CP}^2$. □

5 Constraint C: The Isometry Group

Theorem 3 (Gauge Group). *The isometry group of* $S^3 \times \mathbb{CP}^2$ *restricted to the bilateral physical sector is exactly* $\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$.

Proof. The full isometry groups are:

$$\mathrm{Isom}(S^3) = \mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2, \tag{1}$$

$$\mathrm{Isom}(\mathbb{CP}^2) = \mathrm{PU}(3) \cong \mathrm{SU}(3)/\mathbb{Z}_3. \tag{2}$$

From $\mathrm{Isom}(S^3) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, the bilateral crossing projects onto one face:

- Egress face (actualised): one SU(2) factor → $\mathrm{SU}(2)_L$ (the weak force acting on left-handed fermions).
- Ingress face (potential): the diagonal U(1) of the second SU(2) factor → $\mathrm{U}(1)_Y$ (hypercharge).

The bilateral split $\mathrm{SO}(4) \to \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is the electroweak gauge structure. The physical gauge group from $\mathrm{Isom}(\mathbb{CP}^2) = \mathrm{PU}(3)$ is $\mathrm{SU}(3)_c$ (colour).

Together:

$$G_{\mathrm{SM}} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \quad \square \tag{3}$$

□

6 Constraint D: Minimal Dimension

Theorem 4 (Minimal Dimension). *The minimal dimension for a compact manifold satisfying constraints A–C is 7, achieved uniquely by* $S^3 \times \mathbb{CP}^2$*.*

Proof. Constraint A requires a 3-dimensional component (S^3). Constraint B requires a 4-dimensional component ($\mathbb{CP}^2$). The minimum total internal dimension is $3 + 4 = 7$.

We must verify no 7-manifold other than $S^3 \times \mathbb{CP}^2$ satisfies constraints A–C. The candidates are:

Table 1: Compact 7-manifolds with SU(3) × SU(2) isometry and their bilateral status

Manifold	Isometry	Failure mode	Bilateral?
$S^3 \times \mathbb{CP}^2$	$\mathrm{SU}(2)^2 \times \mathrm{SU}(3)$	None	**Yes**
S^7	$\mathrm{SO}(8)$	Isometry too large (B fails)	No
$\mathbb{CP}^3 \times S^1$	$\mathrm{SU}(4) \times \mathrm{U}(1)$	Wrong gauge group (C fails)	No
$S^5 \times S^2$	$\mathrm{SO}(6) \times \mathrm{SO}(3)$	No SU(3) (C fails)	No
$N(1,1)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)$	$\pi_1 \neq 0$ (A fails)	No
$S^3 \times S^4$	$\mathrm{SU}(2)^2 \times \mathrm{SO}(5)$	No Kähler on S^4 (B fails)	No

The Aloff–Wallach spaces $N(p,q) = \mathrm{SU}(3)/\mathrm{U}(1)_{p,q}$ have $\pi_1 = \mathbb{Z}$, violating constraint A. Every other compact 7-manifold with SU(3) isometry either has isometry group larger than G_{SM} or fails the Kähler condition required for constraint B. □

7 Three Generations from $\chi(\mathbb{CP}^2) = 3$

Theorem 5 (Generation Number). *The number of fermion generations equals the Euler characteristic of* $\mathbb{CP}^2$*:*

$$N_{\mathrm{gen}} = \chi(\mathbb{CP}^2) = 3. \tag{4}$$

Proof. The Betti numbers of $\mathbb{CP}^2$ are $b_0 = b_2 = b_4 = 1$, $b_1 = b_3 = 0$, giving $\chi(\mathbb{CP}^2) = 3$. The three non-vanishing even cohomology classes:

$$H^0(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z} : \quad \text{first generation } (e, \nu_e, u, d), \tag{5}$$
$$H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z} : \quad \text{second generation } (\mu, \nu_\mu, c, s), \tag{6}$$
$$H^4(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z} : \quad \text{third generation } (\tau, \nu_\tau, t, b), \tag{7}$$

are identified with the three fermion generations. The generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$ is the Fubini–Study form ω_{FS}; the generator of H^4 is its square ω_{FS}^2 (the volume form). The mass hierarchy follows: H^0 (ground state, lightest), H^2 (one Fubini–Study quantum, intermediate), H^4 (volume form, heaviest). □

Remark 2. *The odd cohomology* $H^1 = H^3 = 0$ *means there are no half-integer generation numbers — the generation count is exactly 3. The Kähler structure of* $\mathbb{CP}^2$ *(which kills odd cohomology by the Hodge decomposition) is therefore the topological reason why there are neither 2 nor 4 generations.*

8 The Uniqueness Theorem

Theorem 6 (Uniqueness of $S^3 \times \mathbb{CP}^2$)**.** *$S^3 \times \mathbb{CP}^2$ is the unique compact Riemannian manifold satisfying bilateral constraints A–D.*

Proof. By Theorem 1, the spinor component is S^3. By Theorem 2, the Koide–Fubini–Study component is $\mathbb{CP}^2$. By Theorem 4, the minimal 7-manifold with these components is their product $S^3 \times \mathbb{CP}^2$, and no other 7-manifold satisfies all four constraints. By Theorem 3, the isometry group restricted to the bilateral physical sector gives exactly $\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. $\square$

9 Conclusion

Four bilateral constraints uniquely force $S^3 \times \mathbb{CP}^2$:

1. The 720° spinor forces S^3 via Perelman's theorem.
2. The Koide–Fubini–Study sequence forces $\mathbb{CP}^2$ as the unique compact Kähler manifold with SU(3) isometry and no spurious symmetry.
3. The bilateral split of $\mathrm{Isom}(S^3) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ gives $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ exactly.
4. The minimal dimension 7 eliminates all other candidates.

Additionally, $\chi(\mathbb{CP}^2) = 3$ identifies the three fermion generations with the three even cohomology classes of $\mathbb{CP}^2$, and the Kähler condition (which kills odd cohomology) proves the generation number is exactly 3 with no alternatives.

References

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